AN INTERACE RELAXATION FOR THE PATCHING OF SPECTRAL COLLOCATION AND FINITE DIFFERENCE SCHEMES

by

Wilhelm Heinrichs
Universität Essen, Ingenieurmathematik (FB 10)
Universitätsstr. 3, D-45117 Essen, Germany
E-Mail: wheinric@ing-math.uni-essen.de

Abstract

Domain decompositions based on patching techniques for combined spectral collocation and finite difference schemes are considered. The methods are applied to elliptic problems. On the interface lines continuity of normal derivatives is required where appropriate interpolation schemes become necessary. An interface relaxation based on a Dirichlet - Neumann iteration is investigated. Our approach is finally applied to a singular perturbation problem. Numerical results are presented which demonstrate the good performance of our patched schemes.

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Figures: 1

Tables: 5

1 Introduction

Spectral methods (see, e.g., Canuto et al. [2] or Gottlieb and Orszag [4], [7]) employ global polynomials for the discretization of elliptic boundary value problems. Hence they give very accurate approximations for smooth solutions with relatively few degrees of freedom. In order to make these methods applicable for complex geometries or local problems (e.g., singularities, layers, adaptive refinements) it makes sense to combine them with local discretizations resulting from finite difference (or finite element) schemes. For elliptic problems we consider the patching of spectral schemes with finite differences. Here we employ a domain decomposition technique originally proposed by Orszag [7]. For this purpose the original problem is decomposed into subproblems where continuity of the normal derivatives on the interface lines is required. Special problems arise where the subdomains are matching. On one subdomain the solution is approximated by a collocation (or pseudospectral) scheme. Here we employ an approximation with Chebyshev polynomials and Chebyshev Gauss-Lobatto nodes. This allows the efficient implementation with Fast Fourier Transforms (FFTs). On the other subdomain a standard second order finite difference scheme with equidistant nodes is employed. Due to the different sets of nodes second order interpolation techniques on the interface line are necessary. We investigate domain decomposition techniques for the iterative solution of the mixed spectral and finite difference patchings. Here we employ a Dirichlet-Neumann interface relaxation (see [3]) which is iterated until continuity of normal derivatives on the interface is achieved. By numerical results the efficiency of this treatment is demonstrated. With a dynamical relaxation parameter convergence is always achieved within a few numbers of iterations. Our approach is further applied to singular perturbation problems where we were able to resolve boundary layers.

The paper is organized as follows. First we introduce the spectral collocation and finite difference discretizations. The patching at the interface line is performed by suitable interpolations. Then the domain decomposition technique based on a Dirichlet-Neumann iteration is investigated. It is also applied to a singular perturbation problem. Numerical results show the efficiency of our treatment.

2 Decomposition of elliptic problem

We consider second-order elliptic problems given by

$$\begin{aligned} -\Delta u &= f \ in \ \Omega = (-a,b) \times (-1,1), && (1)\\ u &= g \ on \ \partial\Omega. && (2) \end{aligned}$$

Here f, g denote given functions defined on $\Omega, \partial\Omega$ and a, b are positive constants. The original domain Ω is decomposed into two subdomains $\Omega_1 = (-a, 0) \times (-1, 1)$ and $\Omega_2 = (0, b) \times (-1, 1)$. The interface line is denoted by $\Gamma = \{0\} \times (-1, 1)$. On Ω_1 we implement a spectral collocation scheme whereas on Ω_2 standard finite differences are used. The Poisson problem (1),(2) can now be decomposed as follows:

$$\begin{aligned} -\Delta u_k &= f \ in \ \Omega_k, && (3)\\ u_k &= g \ on \ \partial\Omega_k \cap \partial\Omega \ for \ k = 1, 2. && (4) \end{aligned}$$

Here u_k defines the restriction of u to Ω_k. For a smooth decomposition we further require C^1 - interface conditions:

$$u_1 = u_2, \quad \frac{\partial}{\partial x} u_1 = \frac{\partial}{\partial x} u_2 \ on \ \Gamma.$$

2.1 Spectral collocation

For the spectral approximation we introduce the polynomial subspace

$$\mathbb{P}_N = \{Polynomials \ of \ degree \ \leq N \ in \ both \ x \ and \ y\}.$$

The function u_1 is approximated by polynomials $u_{1,N} \in \mathbb{P}_N$. Furthermore we have to introduce the standard Gauss-Lobatto-Chebyshev collocation nodes. They are given by

$$(x_i, y_j) = \left(\cos\frac{i\pi}{N}, \cos\frac{j\pi}{N}\right), \ i, j = 0, \ldots, N,$$

The collocation nodes in Ω_1 are denoted by (x_i^1, y_j^1) where

$$x_i^1 = \frac{a}{2}\left(x_i - 1\right), \ y_j^1 = y_j.$$

The collocation (or pseudo spectral) scheme for (3-4) now reads as follows: Find functions $u_{1,N} \in \mathbb{P}_N$ such that

$$-\Delta u_{1,N}(x_i^1, y_j^1) = f(x_i^1, y_j^1) \ for \ i,j = 1,\ldots,N-1, \tag{5}$$

$$u_{1,N}(x_i^1, y_j^1) = g(x_i^1, y_j^1) \ for \ (x_i^1, y_j^1) \in \partial\Omega_k \cap \partial\Omega. \tag{6}$$

The interface conditions are formulated after the finite difference discreization is introduced.

In the following we write the spectral derivatives in matrix notation. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrix:

$$T = \left(\cos(k\frac{i\pi}{N})\right), \ i,k = 0,\ldots,N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitely given by $\hat{D} = (d_{i,j}) \in \mathbb{R}^{N+1,N+1}$ with

$$d_{i,j} = \begin{cases} \frac{2j}{c_i}, \ j = i+1, i+3, \ldots, N \\ 0\,, \qquad \text{else} \end{cases}$$

and

$$c_i = \begin{cases} 2, \ i = 0, \\ 1, \ \text{else.} \end{cases}$$

Now we are able to write the spectral derivative matrices $D1_{sp}$ and $D2_{sp}$ for the first and second derivatives. They are explicitely given by

$$D1_{sp} = T\hat{D}T^{-1}, \ D2_{sp} = T\hat{D}^2T^{-1}.$$

The spectral operators can be efficiently evaluated by Fast Fourier Transforms (FFTs) in $O(N \log N)$ arithmetic operations. We further introduce the identity matrix $I \in \mathbb{R}^{N+1,N+1}$. By tensor product representation

$$A \otimes B = \left(Ab_{i,j}\right)_{i,j}$$

we are now able to write the spectral Laplacian

$$\Delta_{sp} = D2_{sp} \otimes I + I \otimes D2_{sp}.$$

Partial derivatives can be discretized as follows:

$$\frac{\partial}{\partial x} \cong D1_{sp} \otimes I, \ \frac{\partial}{\partial y} \cong I \otimes D1_{sp}.$$

The 2D spectral operators can be efficiently evaluated by FFTs in $O(N^2 \log N)$ arithmetic operations.

2.2 Finite difference scheme

In Ω_2 standard second order finite differences are employed. Equidistant nodes $(x_i^2, y_j^2), \ i,j = 0,\ldots,M$ where

$$x_i^2 = \frac{b}{2} ih, \ y_j^2 = -1 + jh, \ h = \frac{2}{M}$$

are used for the finite difference scheme. In the 1D-case the second derivative is approximated by central finite differences:

$$u''(x_i) \cong (u(x_{i-1}) - 2u(x_i) + u(x_{i+1}))\,/h^2.$$

By tensor product representation the Laplacian can be approximated as before. The finite difference approximation is denoted by $u_{2,M}$ The discretization is second order accurate. The C^1 - interface conditions are discretized by one-sided (second-order) finite differences. The interface conditions read as follows:

$$\begin{aligned} u_{1,N}(0, y_j^1) &= u_{2,M}(0, y_j^1), \ j = 1,\ldots,N-1, \\ \frac{\partial}{\partial x} u_{1,N}(0, y_j^2) &= \left(-\frac{1}{2} u_{2,M}(x_2^2, y_j^2) + 2u_{2,M}(x_1^2, y_j^2) - \frac{3}{2} u_{2,M}(0, y_j^2)\right)/h \end{aligned}$$

for $j = 1,\ldots,M-1$. Clearly, on the interface line interpolations become necessary. Here we first consider the interpolation from $\{y_j^2\}$ to $\{y_j^1\}$. Let $1 \le i_0 \le N-1$ be fixed such that

$$y_j^2 \le y_{i_0}^1 \le y_{j+2}^2 \ for \ 0 \le j \le M-2.$$

Then the second-order interpolation reads as follows:

$$u_1(0, y^1_{i_0}) \cong w_1 u_2(0, y^2_j) + w_2 u_2(0, y^2_{j+1}) + w_3 u_2(0, y^2_{j+2}),$$

where

$$\begin{aligned} w_1 &= \frac{(y^1_{i_0} - y^2_{j+1})(y^1_{i_0} - y^2_{j+2})}{(y^2_j - y^2_{j+1})(y^2_j - y^2_{j+2})}, \\ w_2 &= \frac{(y^1_{i_0} - y^2_j)(y^1_{i_0} - y^2_{j+2})}{(y^2_{j+1} - y^2_j)(y^2_{j+1} - y^2_{j+2})}, \\ w_3 &= \frac{(y^1_{i_0} - y^2_j)(y^1_{i_0} - y^2_{j+1})}{(y^2_{j+2} - y^2_j)(y^2_{j+2} - y^2_{j+1})}. \end{aligned}$$

For the interpolation from $\{y^1_j\}$ to $\{y^2_j\}$ spectral interpolation is used. Here the polynomial $\frac{\partial}{\partial x} u_{1,N}$ is simply evaluated at the equidistant nodes $(0, y^2_j)$. For this purpose an inverse Fast Fourier Transform is applied on the interface line. Then the resulting polynomial is evaluated at the above equidistant nodes.

3 Interface relaxation

We are now interested in an interface relaxation between the subdomains Ω_1 and Ω_2. At the interface Γ information is exchanged and the iteration is continued until continuity is reached. In one direction Dirichlet information is transferred and in the other direction Neumann information. We use an interface relaxation as proposed in [3], i.e., we solve a sequence of Dirichlet - Neumann problems. We iterate until the error at the interface Γ is smaller than 10^{-8}. The number of iterations required is denoted by IT_{DD}. We start the iteration with initial approximations $u^0_1 = u^0_2 = 0$ and iterate as follows:

$$\begin{aligned} -\Delta u^m_1 &= f \ in \ \Omega_1, \\ u^m_1 &= g \ on \ \partial\Omega_1 \cap \partial\Omega, \\ u^m_1 &= \delta^{m-1} u^{m-1}_2 + (1 - \delta^{m-1}) u^{m-1}_1 \ on \ \Gamma \end{aligned}$$

and

$$
\begin{aligned}
-\Delta u_2^m &= f \; in \; \Omega_2, \\
u_2^m &= g \; on \; \partial\Omega_2 \cap \partial\Omega \\
\frac{\partial}{\partial x} u_2^m &= \frac{\partial}{\partial x} u_1^m \; on \; \Gamma
\end{aligned}
$$

for $m = 1, 2, \ldots$. The discretization is performed as before. δ^m denotes a relaxation parameter which is chosen dynamically in order to accelerate the convergence. The dynamical choice is based on a minimization of the L^2 - error at the interface. The unique number $\delta^m = \delta$ minimizes $||z_m(\delta) - z_{m-1}(\delta)||_2^2$ where $z_m(\delta) = \delta u_2^m + (1-\delta)u_1^m$. δ^m is calculated by

$$
\delta^m = \frac{(e_1^m, e_1^m - e_2^m)}{||e_1^m - e_2^m||_2^2},
$$

where $(.,.)$ denotes the discrete L_2 inner product and

$$
e_i^m = u_i^m - u_i^{m-1} \quad on \; \Gamma \; for \; i = 1, 2
$$

is the difference of two consecutive iterates. Here once more second-order interpolation of e_2^m to the interface nodes of Ω_1 becomes necessary. The parameter δ^m should be located in $(0, 1]$. We numerically calculated the discrete L^2 - error $E2_1, E2_2$ on the domains Ω_1, Ω_2. $E2_1$ is defined by

$$
E2_1 = \left(\sum_{i,j=0}^{N} (u_1 - u_{1,N})^2 (x_i^1, y_j^1) \right)^{1/2} / N.
$$

$E2_2$ is defined in the same manner. We consider examples with a smooth solution given by

$$
u(x, y) = \sin(\pi x) \sin(\pi y) \tag{7}
$$

and interval lenghts

$$
a = 1, b = 1; \;\; a = 2, b = 1; \;\; a = 1, b = 2.
$$

Furthermore we consider a non-smooth solution given by

$$
u(x, y) = \frac{\sin(\pi x)}{x + \delta} \sin(\pi y) \tag{8}
$$

where we choose $\delta = 1.1$ and $a = b = 1$. The singularity occurs near the left boundary but outside the domain Ω. In the tables I-IV we present our numerical results for different values of N, M. Precisely, we consider discretizations with

$$N = 4, M = 8; \;\; N = 8, M = 16; \;\; N = 16, M = 32; \;\; N = 32, M = 64.$$

From our numerical results we observe the expected second-order convergence of the patched scheme. Clearly, on Ω_1 the accuracy is somewhat better than on Ω_2. Globally the accuracy is dominated by the lower order (finite difference) scheme. We always reached convergence after a few steps of the above interface relaxation.

4 Singular perturbation problem

Here we apply the above domain decomposition technique to the singularly perturbed problem

$$\begin{aligned} -\epsilon\Delta u + u_x &= f \ in \ \Omega = (-1,1)^2, && (9) \\ u &= g \ on \ \partial\Omega && (10) \end{aligned}$$

where $\epsilon > 0$ denotes a small perturbation parameter. These equations usually lead to boundary layers. We consider data where the boundary layer occurs near the right boundary at $x = 1$ and $u_x(1, \cdot)$ behaves as $O(\epsilon^{-1})$ for $\epsilon \to 0$. In order to resolve this layer we split the original domain $\Omega = (-1, 1)^2$ into the two subdomains

$$\Omega_1 = (-1, 1 - c\epsilon) \times (-1, 1), \;\; \Omega_2 = (1 - c\epsilon, 1) \times (-1, 1)$$

where c denotes a positive constant. Now the interface line is located in $x = 1 - c\epsilon$. Different choices of c are investigated. A similar decomposition was already successfully applied to the patching of spectral schemes (see [5], [6]). The above system (9),(10) can now be decomposed as follows:

$$\begin{aligned} -\epsilon\Delta v + v_x &= f \ in \ \Omega_1, \\ v &= g \ on \ \partial\Omega_1 \cap \partial\Omega, \\ v(1 - c\epsilon, \cdot) &= w(1 - c\epsilon, \cdot) \ in \ (-1, 1), \end{aligned}$$

$$
\begin{aligned}
v_x(1-c\epsilon,\cdot) &= w_x(1-c\epsilon,\cdot)\ in\ (-1,1),\\
w &= g\ on\ \partial\Omega_2\cap\partial\Omega,\\
-\epsilon\Delta w + w_x &= f\ in\ \Omega_2
\end{aligned}
$$

where $v = u|_{\Omega_1},\ w = u|_{\Omega_2}$. Here we require a C^1-continuity condition at the interface line. In order to apply spectral methods we transform the system into $\tilde{\Omega} = (-2,2)\times(-1,1)$ and split it into the two subdomains

$$
\tilde{\Omega}_1 = (-2,0)\times(-1,1),\ \tilde{\Omega}_2 = (0,2)\times(-1,1).
$$

By introducing a new variable $\tilde{x}\in[-2,2]$ the transformations are given by:

$$
\begin{aligned}
x^- &: [-2,0]\to[-1,1-c\epsilon]\\
x^-(\tilde{x}) &= \frac{1}{2}\left[(2-c\epsilon)(\tilde{x}+1)-c\epsilon\right]
\end{aligned}
$$

and

$$
\begin{aligned}
x^+ &: [0,2]\to[1-c\epsilon,1]\\
x^+(\tilde{x}) &= \frac{1}{2}\left[c\epsilon(\tilde{x}-1)-c\epsilon+2\right].
\end{aligned}
$$

Now we define functions $\tilde{v}$ and $\tilde{w}$ defined on $[-2,0]\times[-1,1]$ and $[0,2]\times[-1,1]$:

$$
\tilde{v}(\tilde{x},y) = v(x^-(\tilde{x}),y),\ \tilde{w}(\tilde{x},y) = w(x^+(\tilde{x}),y).
$$

The functions f,g are transformed in the same way. The above system can now equivalently be written as:

$$
\begin{aligned}
-\epsilon\tilde{v}_{\tilde{x}\tilde{x}} - \frac{\epsilon(2-c\epsilon)^2}{4}\tilde{v}_{yy} + \frac{2-c\epsilon}{2}\tilde{v}_{\tilde{x}} &= \frac{(2-c\epsilon)^2}{4}\tilde{f}\ in\ \tilde{\Omega}_1,\\
\tilde{v} &= \tilde{g}\ on\ \partial\tilde{\Omega}_1\cap\partial\tilde{\Omega},\\
\tilde{v}(0,\cdot) &= \tilde{w}(0,\cdot)\ in\ (-1,1),\\
\frac{2}{2-c\epsilon}v_{\tilde{x}}(0,\cdot) &= \frac{2}{c\epsilon}w_{\tilde{x}}(0,\cdot)\ in\ (-1,1),\\
\tilde{w} &= \tilde{g}\ on\ \partial\tilde{\Omega}_2\cap\partial\tilde{\Omega},\\
-\tilde{w}_{\tilde{x}\tilde{x}} - \frac{c^2\epsilon^2}{4}\tilde{w}_{yy} + \frac{c}{2}\tilde{w}_{\tilde{x}} &= \frac{c^2\epsilon}{4}\tilde{f}\ in\ \tilde{\Omega}_2.
\end{aligned}
$$

Due to our transform the equation in $\tilde{\Omega}_2$ is no more singularly perturbed. For this reason we employ standard (second order) central finite differences for the discretization of $\tilde{w}_{\tilde{x}}$. Low order upwind schemes can be avoided. Stabilization techniques are still needed for the equation in $\tilde{\Omega}_1$. For more details about the problems of stability for singular perturbation problems with spectral methods we refer to the introducing paper of Canuto [1]. Here we employ a special stabilization technique proposed by Heinrichs in [5]. In Ω_1 equation (9) is replaced by the stabilized version

$$-\tilde{\epsilon} u_{xx} - \epsilon u_{yy} + u_x = f + (\epsilon - \tilde{\epsilon}) f_x \ in\ \Omega_1,$$

where $\tilde{\epsilon} = 1/N$ introduces artificial viscosity. A similar technique was used for the interface condition. By neglecting $O(\epsilon)$ terms we obtain

$$(\tilde{\epsilon} - \epsilon) u_x = (\tilde{\epsilon} - \epsilon)(f + \epsilon f_x).$$

In particular, in the smooth part of the solution (away from the boundary layer) we expect the usual high spectral accuracy. Once more we employ the above interface relaxation for the iterative solution of the domain decomposition problem. We start with $\tilde{v}^0 = \tilde{w}^0 = 0$ and iterate as follows:

$$\begin{aligned} -\tilde{\epsilon}\tilde{v}^m_{\tilde{x}\tilde{x}} - \frac{\epsilon(2-c\epsilon)^2}{4}\tilde{v}^m_{yy} + \frac{2-c\epsilon}{2}\tilde{v}^m_{\tilde{x}} &= \frac{(2-c\epsilon)^2}{4}\left(\tilde{f} + \frac{2\epsilon}{2-c\epsilon}\tilde{f}_{\tilde{x}}\right) \ in\ \tilde{\Omega}_1, \\ \tilde{v}^m &= \tilde{g} \ on\ \partial\tilde{\Omega}_1 \cap \partial\tilde{\Omega}, \\ \tilde{v}^m(0,\cdot) &= \delta^{m-1}\tilde{w}^{m-1}(0,\cdot) + (1-\delta^{m-1})\tilde{v}^{m-1}(0,\cdot) \end{aligned}$$

and

$$\begin{aligned} -\tilde{w}^m_{\tilde{x}\tilde{x}} - \frac{c^2\epsilon^2}{4}\tilde{w}^m_{yy} + \frac{c}{2}\tilde{w}^m_{\tilde{x}} &= \frac{c^2\epsilon}{4}\tilde{f} \ in\ \tilde{\Omega}_2, \\ \tilde{w}^m &= \tilde{g} \ on\ \partial\tilde{\Omega}_2 \cap \partial\tilde{\Omega}, \\ w^m_{\tilde{x}}(0,\cdot) &= \frac{c\tilde{\epsilon}}{2-c\epsilon} v^m_{\tilde{x}}(0,\cdot) + c\frac{\epsilon-\tilde{\epsilon}}{2}\left(\tilde{f}(0,\cdot) + \frac{2\epsilon}{2-c\epsilon}\tilde{f}_{\tilde{x}}(0,\cdot)\right) \end{aligned}$$

for $m = 1, 2, \ldots$. The relaxation parameter δ^m is determined in the same manner as before. We consider an example where the data f, g are chosen such that the exact solution is given by:

$$u(x, y) = \phi(x)\psi(y) \tag{11}$$

where

$$\phi(x) = \frac{e^{\frac{x-1}{\epsilon}} - e^{-\frac{2}{\epsilon}}}{1 - e^{-\frac{2}{\epsilon}}}, \ \psi(y) = (1 - y^2).$$

The right hand side f results in

$$f(x,y) = 2\epsilon\phi(x).$$

In our numerical experiments we consider $\epsilon = 10^{-6}$. It turned out that c should slightly increase in N. In table V we present our numerical results. Once more the approximation error is dominated by the second order finite difference scheme. The boundary layer is fairly resolved.

Conclusion

The patching of spectral and finite difference schemes is considered. On the interface lines second-order interpolations are required. An interface relaxation for the iterative solution over the spectral and finite difference subdomains is successfully implemented. This domain decomposition technique is further applied to singular perturbation problems where we were able to resolve boundary layers. Numerical results are presented which show the convergence properties of our approach.

FIGURE

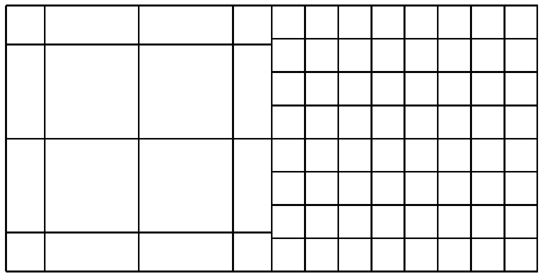

Figure I. The grids for $N = 4$ and $M = 8$.

TABLES

N	M	$E2_1$	$E2_2$	IT_{DD}
4	8	$1.18{\cdot}10^{-1}$	$3.26{\cdot}10^{-1}$	6
8	16	$1.02{\cdot}10^{-2}$	$3.59{\cdot}10^{-2}$	6
16	32	$3.35{\cdot}10^{-3}$	$1.25{\cdot}10^{-2}$	13
32	64	$1.15{\cdot}10^{-3}$	$4.38{\cdot}10^{-3}$	15

Table I. Numerical results for example (7), $a = 1, b = 1$.

N	M	$E2_1$	$E2_2$	IT_{DD}
4	8	$1.10{\cdot}10^{-1}$	$1.73{\cdot}10^{-1}$	7
8	16	$8.28{\cdot}10^{-3}$	$3.49{\cdot}10^{-2}$	8
16	32	$2.87{\cdot}10^{-3}$	$1.25{\cdot}10^{-2}$	12
32	64	$9.72{\cdot}10^{-4}$	$4.38{\cdot}10^{-3}$	15

Table II. Numerical results for example (7), $a = 2, b = 1$.

N	M	$E2_1$	$E2_2$	IT_{DD}
4	8	$1.64{\cdot}10^{-1}$	$4.08{\cdot}10^{-1}$	7
8	16	$3.07{\cdot}10^{-2}$	$6.53{\cdot}10^{-2}$	8
16	32	$1.03{\cdot}10^{-2}$	$2.22{\cdot}10^{-2}$	14
32	64	$3.54{\cdot}10^{-3}$	$7.72{\cdot}10^{-3}$	15

Table III. Numerical results for example (7), $a = 1, b = 2$.

N	M	$E2_1$	$E2_2$	IT_{DD}
4	8	$2.19 \cdot 10^{-1}$	$2.28 \cdot 10^{-1}$	6
8	16	$1.23 \cdot 10^{-2}$	$1.25 \cdot 10^{-2}$	6
16	32	$1.56 \cdot 10^{-3}$	$6.78 \cdot 10^{-3}$	13
32	64	$6.32 \cdot 10^{-4}$	$2.53 \cdot 10^{-3}$	15

Table IV. Numerical results for example (8).

N	M	c	$E2_1$	$E2_2$	IT_{DD}
4	8	$N/2 = 2.0$	$2.68 \cdot 10^{-2}$	$1.46 \cdot 10^{-2}$	5
8	16	$N/2 = 4.0$	$6.00 \cdot 10^{-3}$	$5.93 \cdot 10^{-3}$	7
16	32	$N/3 = 5.\overline{3}$	$3.22 \cdot 10^{-3}$	$2.84 \cdot 10^{-3}$	12
32	64	$N/5 = 6.4$	$1.58 \cdot 10^{-3}$	$1.27 \cdot 10^{-3}$	12

Table V. Numerical results for example (11).

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