CONVERGENCE OF CHEBYSHEV SPECTRAL APPROXIMATIONS FOR EIGENVALUES AND EIGENFUNCTIONS OF DISCONTINUOUS PROBLEMS

by

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Abstract

For a second order model problem with discontinuous coefficients the convergence of Chebyshev spectral schemes is investigated. We consider Chebyshev Galerkin, Tau and collocation methods. It is numerically shown that the Galerkin and Tau methods are third order accurate for the eigenvalues and of order 2.5 for the eigenfunctions. For the collocation method with pointwise evaluation of the discontinuous coefficients we obtained only second order accuracy. This could be improved by using a truncated Chebyshev series of the coefficients. Numerical results are presented which show the convergence rates for the different spectral schemes.

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Tables: 4
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1 Introduction

We consider second order eigenvalue problems with discontinuous coefficients. These problems are motivated from applications in nonlinear optics. When different media are considered, the coefficients are only piecewise smooth and the accuracy is lost. Min and Gottlieb [4] already studied the convergence of the Fourier approximation for such kind of problems. The surprising fact was that the order of convergence of the the eigenvalues obtained by the Fourier-Galerkin is cubic. When the Fourier collocation method is applied, the results are only second order accurate. These results were proven by using the minmax principle [2]. For the collocation method it turned out that by preprocessing the discontinuous coefficients the accuracy could be improved. By using the point values of the truncated Fourier series of the coefficients, one recovers third order accuracy for the eigenvalues and order 2.5 for the eigenfunctions. For the first five eigenvalues this was supported by numerical computations.

Here we consider second order eigenvalue problems with discontinuous coefficients and Dirichlet boundary conditions. We use different Chebyshev spectral schemes (see [1, 3]) for the approximation of the eigenvalues and eigenfunctions. These methods have the advantage (in compare with Legendre methods) that Fast Fourier Transforms (FFTs) can be employed. For the Galerkin and Tau method we obtain third order accuracy for the eigenvalues and order 2.5 for the eigenfunctions. For the collocation method the results are only second order. By preprocessing the discontinuous coefficients (truncated Chebyshev series) we were also able to recover the third order accuracy for the eigenvalues and order 2.5 for the eigenfunctions. This is in full agreement with the Fourier case and was confirmed by numerical computations for the first five eigenvalues.

The paper is organized as follows. In section 2, we present the problem and give an analytical computation of the eigenvalues and eigenfunctions. In section 3, we introduce the different spectral schemes for the approximation of the problem. Finally we present in section 4 our numerical results which confirm the theoretical predictions from the Fourier case.

2 The discontinuous eigenvalue problem

We consider the following eigenvalue problem with a piecewise constant coefficient

$$
\begin{aligned}
-\frac{d^2u}{dx^2} &= \lambda\epsilon(x)u \quad \text{for } x \in (-1,1), && (1)\\
u(-1) &= u(1) \;=\; 0. && (2)
\end{aligned}
$$

The coefficient ϵ is explicitly given by

$$
\epsilon \;=\; \begin{cases} 1\,, & -1 < x < 0, \\ \beta^2, & 0 \leq x < 1,\;\; \beta \neq 1, \end{cases}
$$

where a jump in $x = 0$ occurs. In the practical computations we consider the case $\beta = 2$. The eigenfunction $u_l(x)$ due to the given Dirichlet boundary conditions is given by

$$
u_l(x) \;=\; \begin{cases} A\sin\sqrt{\lambda_l}(x+1)\;, & -1 < x < 0, \\ B\sin\beta\sqrt{\lambda_l}(x-1), & 0 \leq x < 1, \end{cases} \tag{3}
$$

where the constants A, B and the eigenvalue λ_l are determined such that the system resulting from the regularity conditions in $x = 0$:

$$
\begin{aligned}
A\sin\sqrt{\lambda} + B\sin\beta\sqrt{\lambda} &= 0\\
A\sqrt{\lambda}\cos\sqrt{\lambda} - B\beta\sqrt{\lambda}\cos\beta\sqrt{\lambda} &= 0
\end{aligned}
$$

has a nontrivial solution. By computing the determinant of the above system and setting it to zero we derive

$$
\sin(\beta+1)\sqrt{\lambda} \;+\; (\beta-1)\sin\sqrt{\lambda}\;\cos\beta\sqrt{\lambda} = 0.
$$

Considering the case $\beta = 2$, the eigenvalues λ are determined by

$$
\sin 3\sqrt{\lambda} \;+\; \sin\sqrt{\lambda}\;\cos 2\sqrt{\lambda} = 0.
$$

By setting $y = \sin\sqrt{\lambda}$ and using addition theorems we obtain the characteristic equation

$$y(6y^2 - 4) = 0.$$

Hence there are families of eigenvalues given by

$$\sin\sqrt{\lambda} = 0,\ \pm\sqrt{\frac{2}{3}}\ .$$

From this we deduce the values $\sqrt{\lambda}$ as

$$\begin{aligned}\sqrt{\lambda} &= k\pi,\ k = 1, 2, \ldots,\ \mathit{or} \\ \sqrt{\lambda} &= \alpha, \pi - \alpha, \pi + \alpha, 2\pi - \alpha \quad + [\ 2(k-1)\pi]\,,\ k = 1, 2, \ldots.\end{aligned}$$

where $\alpha = \arcsin\sqrt{\frac{2}{3}} \cong 0.95531662$. Hence for $k = 1$ we obtain the five smallest values

$$\sqrt{\lambda} = \alpha, \pi - \alpha, \pi, \pi + \alpha, 2\pi - \alpha$$

and corresponding eigenvalues

$$\lambda = \alpha^2, (\pi - \alpha)^2, \pi^2, (\pi + \alpha)^2, (2\pi - \alpha)^2$$

which are approximately given by

$$\begin{aligned}\lambda_1 &\cong 0.912629841, \\ \lambda_2 &\cong 4.779802903, \\ \lambda_3 &\cong 9.869604401, \\ \lambda_4 &\cong 16.78466558, \\ \lambda_5 &\cong 28.38618477.\end{aligned}$$

Furthermore we are interested in the eigenfunctions. In the case $\sqrt{\lambda} = k\pi,\ k = 1, 2, \ldots$ the above linear system results in the equation

$$A\cos k\pi - 2B\cos 2k\pi = 0,$$

which leads to

$$B = \begin{cases} A/2 \ , \ k \ even, \\ -A/2, \ k \ \ odd. \end{cases}$$

By fixing $A = 1$ we obtain the normalized eigenfunctions

$$u_k^1(x) = \begin{cases} \sin k\pi(x+1) \quad , & -1 < x < 0, \\ \frac{1}{2}\sin 2k\pi(x-1) \ , & 0 \le x < 1, \ k \ even, \\ -\frac{1}{2}\sin 2k\pi(x-1), & 0 \le x < 1, \ k \ odd, \end{cases}$$

which are determined up to a constant factor. Next we consider the case $\sin\sqrt{\lambda} = \pm\sqrt{\frac{2}{3}}$ and $\cos\sqrt{\lambda} = -\frac{1}{\sqrt{3}}$, *i.e.* $\sqrt{\lambda} = \pi \pm \alpha$, which leads to the equation

$$A - \frac{2}{\sqrt{3}}B = 0$$

and the normalized eigenfunctions

$$u_k^2(x) = \begin{cases} \sin\sqrt{\lambda}(x+1) \quad , & -1 < x < 0, \\ \frac{\sqrt{3}}{2}\sin 2\sqrt{\lambda}(x-1), & 0 \le x < 1, \end{cases}$$

In the case $\sin\sqrt{\lambda} = \pm\sqrt{\frac{2}{3}}$ and $\cos\sqrt{\lambda} = \frac{1}{\sqrt{3}}$, *i.e.* $\sqrt{\lambda} = \alpha, 2\pi - \alpha$, a similar computation yields the eigenfunctions

$$u_k^3(x) = \begin{cases} \sin\sqrt{\lambda}(x+1) \quad , & -1 < x < 0, \\ -\frac{\sqrt{3}}{2}\sin 2\sqrt{\lambda}(x-1), & 0 \le x < 1. \end{cases}$$

A plot of the eigenfunctions belonging to the first five eigenvalues is given in the figure.

3 Chebyshev spectral approximation

Here we consider the Chebyshev spectral approximation with the Galerkin, Tau and collocation method. The collocation method is modified so that a higher order method can be achieved.

3.1 The Galerkin method

First we introduce the weighted inner products

$$(u,v)_w = \int_{-1}^{1} u(x)v(x)w(x)dx$$

and

$$(u,v)_\epsilon = \int_{-1}^{1} \epsilon(x)u(x)v(x)w(x)dx,$$

where w denotes the Chebyshev weight function given by

$$w(x) = \frac{1}{\sqrt{1-x^2}}.$$

Let $\mathbb{P}_N^0$ denote the space of polynomials of degree $\leq N$ which vanish in ± 1:

$$\mathbb{P}_N^0 = \{p \in \mathbb{P}_N : \ p(\pm 1) = 0\}.$$

Now the Galerkin method looks for scalars λ_N and polynomials $u_N \in \mathbb{P}_N^0$ such that

$$-(u_N'', v_N)_w = \lambda_N (u_N, v_N)_\epsilon \quad for\ all\ v_N \in \mathbb{P}_N^0. \tag{4}$$

In the practical computation we choose an approximation with Chebyshev polynomials $T_k(x) = \cos k \arccos x$, i.e.,

$$u_N(x) = \sum_{k=0}^{N-2} a_k (1-x^2) T_k(x). \tag{5}$$

The test functions v_N are the polynomials $(1-x^2)T_l(x),\ l = 0, \ldots, N-2$.

By using the abbreviation $\phi_k = (1-x^2)T_k$ the Galerkin approximation u_N is now determined by the equation

$$-\sum_{k=0}^{N-2} a_k(\phi_k'', \phi_l)_w = \lambda_N \sum_{k=0}^{N-2} a_k(\phi_k, \phi_l)_\epsilon \quad for\ l = 0, \ldots, N-2. \tag{6}$$

Now by introducing the matrices

$$A = ((\phi_k'', \phi_l)_w)_{k,l=0,\ldots,N-2} \in \mathrm{IR}^{N-1,N-1}$$

and

$$B = ((\phi_k, \phi_l)_\epsilon)_{k,l=0,\ldots,N-2} \in \mathrm{IR}^{N-1,N-1}$$

the eigenvalue problem resulting from the Galerkin method reads as

$$Aa_N = \lambda_N B a_N \tag{7}$$

where $a_N = (a_k)_{k=0,\ldots,N-2} \in \mathrm{IR}^{N-1}$. The inner products in A can be exactly evaluated by means of Gauss-Lobatto quadrature with the Gauss-Lobatto nodes $x_i = \cos(i\pi/N),\ i = 0, \ldots, N$. The inner products in B are explicitly calculated. By using the substitution $x = \cos y$ we derive

$$\begin{aligned}(\phi_k, \phi_l)_\epsilon &= \int_0^\pi \epsilon(\cos y) \sin^4 y \cos ky \cos ly dy \\ &= 4\int_0^{\pi/2} \sin^4 y \cos ky \cos ly dy + \int_{\pi/2}^{\pi} \sin^4 y \cos ky \cos ly dy.\end{aligned}$$

The latter integrals are evaluated by using

$$\int \sin^4 y \cos ky \cos ly dy = \frac{1}{2}\int \sin^4 y \cos(k-l) y dy \ + \frac{1}{2}\int \sin^4 y \cos(k+l) y dy$$

and

$$\begin{aligned}&\int^z \sin^4 y \cos \gamma y dy = \\ &= \frac{1}{16}\left(6\frac{\sin \gamma z}{\gamma} - 4\frac{\sin(2-\gamma)z}{2-\gamma} - 4\frac{\sin(2+\gamma)z}{2+\gamma} + \frac{\sin(4-\gamma)z}{4-\gamma} + \frac{\sin(4+\gamma)z}{4+\gamma}\right).\end{aligned}$$

3.2 The Tau method

The Tau method is quite similar to the Galerkin method but it works with different test functions. It looks for scalars λ_N and polynomials $u_N \in \mathbb{P}_N^0$ such that

$$-(u_N'', v_{N-2})_w = \lambda_N (u_N, v_{N-2})_\epsilon \quad \mathit{for\ all}\ v_{N-2} \in \mathbb{P}_{N-2}. \tag{8}$$

We use the same spectral appproximation u_N as before. The test functions v_{N-2} are the Chebyshev polynomials $T_l(x),\ l = 0, \ldots, N-2$. The spectral Tau approximation u_N is now determined by the equation

$$-\sum_{k=0}^{N-2} a_k (\phi_k'', T_l)_w = \lambda_N \sum_{k=0}^{N-2} a_k (\phi_k, T_l)_\epsilon \quad \mathit{for}\ l = 0, \ldots, N-2. \tag{9}$$

By introducing the matrices

$$A = ((\phi_k'', T_l)_w)_{k,l=0,\ldots,N-2} \in \mathbb{R}^{N-1,N-1}$$

and

$$B = ((\phi_k, T_l)_\epsilon)_{k,l=0,\ldots,N-2} \in \mathbb{R}^{N-1,N-1}$$

the eigenvalue problem can be written as in (7). The inner products in A can be exactly evaluated by means of Gauss-Lobatto quadrature. The inner products in B are explicitly calculated. By using the substitution $x = \cos y$ we obtain

$$\begin{aligned}
(\phi_k, T_l)_\epsilon &= \int_0^\pi \epsilon(\cos y) \sin^2 y \cos ky \cos ly dy \\
&= 4 \int_0^{\pi/2} \sin^2 y \cos ky \cos ly dy + \int_{\pi/2}^{\pi} \sin^2 y \cos ky \cos ly dy.
\end{aligned}$$

The latter integrals are evaluated by using

$$\int \sin^2 y \cos ky \cos ly dy = \frac{1}{2} \int \sin^2 y \cos(k-l) y dy \ + \frac{1}{2} \int \sin^2 y \cos(k+l) y dy$$

and

$$\int^z \sin^2 y \cos \gamma y dy = \frac{1}{4} \left(2 \frac{\sin \gamma z}{\gamma} - \frac{\sin(2-\gamma)z}{2-\gamma} - \frac{\sin(2+\gamma)z}{2+\gamma} \right).$$

3.3 The collocation scheme

The collocation scheme is a pointwise method based on the Chebyshev Gauss-Lobatto nodes given by

$$x_i = \cos\frac{i\pi}{N},\ \ i = 0, \dots, N.$$

Here we look for scalars λ_N and polynomials $u_N \in \mathbb{P}_N^0$ such that

$$-u_N''(x_i) = \lambda_N \epsilon(x_i) u_N(x_i) \quad for\ i = 1, \dots, N-1. \tag{10}$$

Since there is a jump in $x = 0$ we obtained favorable results by using the arithmetic mean value, i.e., by setting $\epsilon(0) = (\beta^2 + 1)/2$. In the following we write the spectral system in matrix notation. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrix:

$$T \ = \ \left(\cos(k\frac{i\pi}{N})\right),\ \ i, k = 0, \dots, N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitly given by $\tilde{D} = (\tilde{d}_{i,j}) \in \mathbb{R}^{N+1,N+1}$ with

$$\tilde{d}_{i,j} \ = \ \begin{cases} \frac{2j}{c_i},\ j = i+1, i+3, \dots, N \\ 0\,, \qquad\qquad \text{else} \end{cases}$$

and

$$c_i \ = \ \begin{cases} 2,\ i = 0, \\ 1,\ \text{else}. \end{cases}$$

Now we are able to write the spectral derivative matrix D^2 for the second derivative. It is explicitly given by

$$D^2 = T\tilde{D}^2 T^{-1} \in \mathbb{R}^{N+1,N+1}.$$

The spectral operator can be efficiently evaluated by Fast Fourier Transforms (FFTs) in $O(N \log N)$ arithmetic operations.

By omitting the first and last row resp. column we obtain the matrix $\hat{D}^2 \in \mathrm{IR}^{N-1,N-1}$ belonging to the homogeneous Dirichlet boundary conditions. Furthermore we introduce

$$E = diag(\epsilon(x_i))_{i=1,\ldots,N-1} \in \mathrm{IR}^{N-1,N-1}.$$

Now the eigenvalue problem based on collocation can be written as

$$\hat{D}^2 \hat{u}_N = \lambda_N E \hat{u}_N, \tag{11}$$

where $\hat{u}_N = (u(x_1), \ldots, u(x_{N-1}))^t$.

3.4 The improved collocation scheme

Similar to the Fourier method (see [4]) we were able to enhance the accuracy by a preprocessing strategy. We expand the discontinuous coefficient ϵ in the finite Chebyshev series represented by

$$\epsilon_N(x) = \sum_{k=0}^{N} \hat{\epsilon}_k T_k(x)$$

where

$$\hat{\epsilon}_k = \frac{(\epsilon, T_k)_w}{(T_k, T_k)_w}.$$

Once more by using the substitution $x = \cos y$ we obtain

$$\begin{aligned}
(\epsilon, T_k)_w &= \int_{-1}^{0} T_k(x) w(x) dx + \beta^2 \int_0^1 T_k(x) w(x) dx \\
&= \int_{\pi/2}^{\pi} \cos ky dy + \beta^2 \int_0^{\pi/2} \cos ky dy \\
&= (\beta^2 - 1) \frac{1}{k} \sin \frac{k\pi}{2}
\end{aligned}$$

Since $(T_k, T_k)_w = \frac{\pi}{2} c_k$ we arrive at

$$\hat{\epsilon}_k = \begin{cases} \frac{1}{2}(\beta^2 + 1), & k = 0, \\ (\beta^2 - 1) \frac{2}{k\pi} \sin \frac{k\pi}{2}, & k \geq 1. \end{cases}$$

As before we introduce the modified matrix

$$E_N = diag(\epsilon_N(x_i))_{i=1,\ldots,N-1} \in \mathrm{IR}^{N-1,N-1}.$$

The eigenvalue problem based on the improved collocation scheme can now be written as

$$\hat{D}^2 \hat{u}_N = \lambda_N E_N \hat{u}_N. \tag{12}$$

4 Numerical results

From the tables I,II we observe that the Galerkin and Tau method yield very similar results. In general the eigenvalues are third order accurate whereas the eigenfunctions are of order 2.5. Only the eigenvalue π^2 is approximated by a fifth order scheme whereas the corresponding eigenfunction is only second order accurate. The standard collocation scheme (see table III) only yields a second order method. This could could be improved by a preprocessing of the discontinuous coefficient. Hereafter the modified collocation scheme (see table IV) was also third order accurate for the eigenvalues and of order 2.5 for the eigenfunctions. Once more for the eigenvalue π^2 we observe a higher order convergence. These results are in full agreement with the theoretical predictions from the Fourier case.

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λ_i	N	λ_i^N	$\frac{\lvert\lambda_i^N-\lambda_i\rvert}{\lambda_i}$	Order	$\lvert\lvert u_i - u_i^N\rvert\rvert_{l_2}$	Order
0.9126298409	16	0.9127158612	9.426(-5)	–	2.190(-4)	–
	32	0.9126390789	1.012(-5)	3.22	4.457(-5)	2.30
	64	0.9126309190	1.181(-6)	3.10	8.434(-6)	2.40
	128	0.9126299713	1.429(-7)	3.05	1.542(-6)	2.45
	256	0.9126298569	1.758(-8)	3.02	2.773(-7)	2.48
	512	0.9126298428	2.149(-9)	3.03	4.943(-8)	2.49
4.7798029033	16	4.7820034319	4.604(-4)	–	1.349(-3)	–
	32	4.7800413238	4.988(-5)	3.21	2.452(-4)	2.46
	64	4.7798314606	5.975(-6)	3.06	4.497(-5)	2.45
	128	4.7798064161	7.349(-7)	3.02	8.137(-6)	2.47
	256	4.7798033394	9.124(-8)	3.01	1.457(-6)	2.48
	512	4.7798029576	1.136(-8)	3.01	2.593(-7)	2.49
9.8696044011	16	9.8700958053	4.979(-5)	–	7.574(-3)	–
	32	9.8696161356	1.189(-6)	5.39	2.011(-3)	1.91
	64	9.8696047472	3.506(-8)	5.08	4.886(-4)	2.04
	128	9.8696044118	1.081(-9)	5.02	1.239(-4)	1.98
	256	9.8696044014	3.299(-11)	5.03	3.074(-5)	2.01
16.784665581	16	16.829430909	2.667(-3)	–	6.516(-3)	–
	32	16.788044737	2.013(-4)	3.73	9.472(-4)	2.78
	64	16.785036579	2.210(-5)	3.19	1.636(-4)	3.19
	128	16.784709867	2.639(-6)	3.07	2.899(-5)	2.50
	256	16.784671013	3.237(-7)	3.03	5.149(-6)	2.50
	512	16.784666254	4.012(-8)	3.01	9.132(-7)	2.50
28.386184768	16	28.581348409	6.875(-3)	–	1.948(-2)	–
	32	28.396195113	3.526(-4)	4.29	1.847(-3)	3.40
	64	28.387245070	3.735(-5)	3.24	2.916(-4)	2.66
	128	28.386310902	4.444(-6)	3.07	5.013(-5)	2.54
	256	28.386200260	5.457(-7)	3.03	8.798(-6)	2.51
	512	28.386186690	6.773(-8)	3.01	1.552(-6)	2.50

Table I. Results for the Chebyshev Galerkin method

λ_i	N	λ_i^N	$\frac{\lvert\lambda_i^N-\lambda_i\rvert}{\lambda_i}$	Order	$\lvert\lvert u_i - u_i^N\rvert\rvert_{l_2}$	Order
0.9126298409	16	0.9127294357	1.091(-4)	–	1.554(-4)	–
	32	0.9126397209	1.083(-5)	3.33	3.875(-5)	2.00
	64	0.9126309549	1.221(-6)	3.15	7.921(-6)	2.29
	128	0.9126299734	1.453(-7)	3.07	1.497(-6)	2.40
	256	0.9126298570	1.772(-8)	3.04	2.733(-7)	2.45
	512	0.9126298431	2.422(-9)	2.87	4.908(-8)	2.48
4.7798029033	16	4.7829654477	6.616(-4)	–	1.107(-3)	–
	32	4.7800820850	5.841(-5)	3.50	2.168(-4)	2.35
	64	4.7798336704	6.437(-6)	3.18	4.236(-5)	2.36
	128	4.7798065460	7.621(-7)	3.08	7.904(-6)	2.42
	256	4.7798033473	9.289(-8)	3.04	1.436(-6)	2.46
	512	4.7798029578	1.141(-8)	3.03	2.574(-7)	2.48
9.8696044011	16	9.8706902710	1.100(-4)	–	7.395(-3)	–
	32	9.8696256694	2.155(-6)	5.67	1.997(-3)	1.89
	64	9.8696049742	5.806(-8)	5.21	4.880(-4)	2.03
	128	9.8696044180	1.709(-9)	5.09	1.238(-4)	1.98
	256	9.8696044017	6.251(-11)	4.77	3.074(-5)	2.01
16.784665581	16	16.844171636	3.545(-3)	–	6.619(-3)	–
	32	16.788325715	2.181(-4)	4.02	8.617(-4)	2.94
	64	16.785050136	2.291(-5)	3.25	1.550(-4)	2.47
	128	16.784710644	2.685(-6)	3.09	2.819(-5)	2.46
	256	16.784671060	3.264(-7)	3.04	5.077(-6)	2.47
	512	16.784666257	4.029(-8)	3.02	9.068(-7)	2.49
28.386184768	16	28.679467771	1.033(-2)	–	2.435(-2)	–
	32	28.397444320	3.967(-4)	4.70	1.757(-3)	3.79
	64	28.387301974	3.936(-5)	3.33	2.790(-4)	2.65
	128	28.386314136	4.557(-6)	3.11	4.887(-5)	2.51
	256	28.386200454	5.526(-7)	3.04	8.680(-6)	2.49
	512	28.386186703	6.817(-8)	3.02	1.541(-6)	2.49

Table II. Results for the Chebyshev Tau method

λ_i	N	λ_i^N	$\frac{\lvert\lambda_i^N-\lambda_i\rvert}{\lambda_i}$	Order	$\lvert\lvert u_i-u_i^N\rvert\rvert_{l_2}$	Order
	16	0.9163082774	4.031(-3)	–	1.560(-3)	–
	32	0.9135809792	1.042(-3)	1.95	4.055(-4)	1.94
0.9126298409	64	0.9128722704	2.656(-4)	1.97	1.036(-4)	1.97
	128	0.9126910757	6.710(-5)	1.98	2.618(-5)	1.98
	256	0.9126452311	1.686(-5)	1.99	6.582(-6)	1.99
	512	0.9126336988	4.227(-6)	2.00	1.650(-6)	2.00
	16	4.7206776245	1.237(-2)	–	9.825(-3)	–
	32	4.7666182686	2.758(-3)	2.17	2.568(-3)	1.94
4.7798029033	64	4.7766792176	6.535(-4)	2.08	6.561(-4)	1.97
	128	4.7790419934	1.592(-4)	2.04	1.659(-4)	1.98
	256	4.7796150853	3.929(-5)	2.02	4.170(-5)	1.99
	512	4.7797562442	9.762(-6)	2.01	1.046(-5)	2.00
	16	9.8721372030	2.566(-4)	–	2.911(-2)	–
	32	9.8696753280	7.186(-6)	5.16	6.800(-3)	2.10
9.8696044011	64	9.8696065556	2.183(-7)	5.04	1.658(-3)	2.04
	128	9.8696044011	6.773(-9)	5.01	4.137(-4)	2.00
	256	9.8696044032	2.129(-10)	4.99	1.032(-4)	2.00
	16	17.023412483	1.422(-2)	–	4.178(-2)	–
	32	16.851359559	3.974(-3)	1.84	9.337(-3)	2.16
16.784665581	64	16.802583050	1.067(-3)	1.90	2.286(-3)	2.03
	128	16.789333019	2.781(-4)	1.94	5.697(-4)	2.01
	256	16.785858353	7.106(-5)	1.97	1.425(-4)	2.00
	512	16.784967177	1.797(-5)	1.98	3.564(-5)	2.00
	16	27.089225733	4.569(-2)	–	6.737(-2)	–
	32	28.158315688	8.027(-3)	2.51	1.568(-2)	2.10
28.386184768	64	28.337200447	1.726(-3)	2.22	3.943(-3)	1.99
	128	28.374745741	4.030(-4)	2.10	9.948(-4)	1.99
	256	28.383415829	9.755(-5)	2.05	2.503(-4)	1.99
	512	28.385503309	2.401(-5)	2.02	6.279(-5)	2.00

Table III. Results for the collocation scheme with pointwise evaluation

λ_i	N	λ_i^N	$\frac{\lvert\lambda_i^N-\lambda_i\rvert}{\lambda_i}$	Order	$\lvert\lvert u_i - u_i^N\rvert\rvert_{l_2}$	Order
	16	0.9127357949	1.161(-4)	–	2.125(-4)	–
	32	0.9126400943	1.123(-5)	3.37	4.267(-5)	2.32
0.9126298409	64	0.9126309776	1.246(-6)	3.17	8.227(-6)	2.37
	128	0.9126299748	1.468(-7)	3.09	1.522(-6)	2.43
	256	0.9126298572	1.790(-8)	3.04	2.755(-7)	2.47
	512	0.9126298429	2.230(-9)	3.00	4.927(-8)	2.48
	16	4.7843174891	9.445(-4)	–	1.665(-3)	–
	32	4.7801145666	6.520(-5)	3.86	2.287(-4)	2.86
4.7798029033	64	4.7798350213	6.720(-6)	3.28	4.284(-5)	2.42
	128	4.7798066150	7.766(-7)	3.11	7.935(-6)	2.43
	256	4.7798033511	9.370(-8)	3.05	1.439(-6)	2.46
	512	4.7798029581	1.147(-8)	3.03	2.577(-7)	2.48
	16	9.8326824305	3.741(-3)	–	1.133(-2)	–
	32	9.8686907151	9.258(-5)	5.34	2.306(-3)	2.30
9.8696044011	64	9.8695604646	4.452(-6)	4.38	5.176(-4)	2.16
	128	9.8696019378	2.496(-7)	4.16	1.275(-4)	2.02
	256	9.8696042547	1.483(-8)	4.07	3.117(-5)	2.03
	512	9.8696043913	9.912(-10)	3.90	7.767(-6)	2.00
	16	17.116185205	1.975(-1)	–	1.797(-1)	–
	32	16.790585675	3.527(-3)	5.81	1.653(-2)	3.44
16.784665581	64	16.785145224	2.858(-4)	3.63	1.815(-3)	3.19
	128	16.784715882	2.997(-5)	3.25	2.857(-4)	2.67
	256	16.784671371	3.449(-6)	3.12	5.022(-5)	2.51
	512	16.784666276	4.142(-7)	3.06	8.980(-6)	2.48
	16	27.177707058	4.257(-1)	–	7.557(-1)	–
	32	28.409666816	8.272(-3)	5.69	3.199(-2)	4.56
28.386184768	64	28.387606835	5.010(-4)	4.05	2.885(-3)	3.47
	128	28.386327349	5.023(-5)	3.32	4.784(-4)	2.59
	256	28.386201150	5.771(-6)	3.12	8.566(-5)	2.48
	512	28.386186743	6.957(-7)	3.05	1.532(-5)	2.48

Table IV. Results for the collocation scheme with truncated Chebyshev series