FIRST-ORDER SYSTEM LEAST-SQUARES FOR THE SPECTRAL APPROXIMATION OF SECOND-ORDER ELLIPTIC PROBLEMS

by

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Abstract

A least squares solver for the pseudo spectral (or collocation) approximation of elliptic equations is presented. The elliptic spectral system is decoupled into a first-order system. For Helmholtz equations and the spectral Pseudo-Laplacian (Stokes equations) it is shown that the condition number is decreased from $O(N^4)$ to $O(N^2)$. By suitable finite difference preconditioning the condition number is further reduced to $\frac{\pi}{2}$. Numerical results are presented which substantiate the usefulness of our concept.

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Tables: 6

1 Introduction

Spectral methods (see, e.g., Canuto et al. [?] or Gottlieb and Orszag [?], [?]) employ global polynomials for the discretization of elliptic boundary value problems. Hence they give very accurate approximations for smooth solutions with relatively few degrees of freedom. A drawback of the spectral approach is due to the fact that the spectral matrices are nonsymmetric and dense. They yield condition numbers growing as $O(N^4)$ for polynomials of degree N in each variable. By earlier approaches (see [?], [?],[?]) we were already able to reduce the condition number to $O(N^2)$. The main idea was the use of polynomials which implicitely contain homogeneous boundary conditions. Here the reduction of the condition number is achieved by splitting the elliptic equation into a first-order system. The discretization is performed by a pseudo spectral (or collocation) scheme based on staggered grids where a combination of both Gauss- and Gauss-Lobatto-Chebyshev nodes is employed. The number of unknowns is increasing from $O(N)$ to $O(2N)$ in 1D and from $O(N^2)$ to $O(3N^2)$ in 2D. Since for spectral schemes the degrees of freedom N are small compared to finite differences or finite elements the extra amount of work is not dramatical. A similar approach was already sucessfully employed for the Stokes problem in streamfunction formulation (see [?],[?]). The biharmonic operator was splitted into a system with Laplace operators. This reduced the condition number from $O(N^8)$ to $O(N^4)$. By using the proposed splitting of the Laplace operator into a system of first-order operators the condition number is further reduced to $O(N^2)$. Hence a dramatical improvement in condition numbers is achieved. By further using finite difference preconditioning it is numericaly shown that the eigenvalues of the preconditioned spectral operator are real and bounded by $\frac{\pi}{2}$. This bound is well known for preconditioning of first-order spectral operators with finite differences (see [?]). The first-order system can be efficiently solved by least-squares solvers (see [?], [? for finite elements). Here we recommend a multigrid solver with line Gauss-Seidel or incomplete LU (ILU) decomposition (see [?], [?]) for the corresponding normal equations. Furthermore our results are extended to the Stokes or Navier-Stokes equations. By a splitting technique (see [?],[?],[?],[?]) based on the Uzawa algorithm the system is decoupled into Helmholtz equations for the velocity components and a Pseudo-Laplacian for the pressure. It is shown how the spectral Pseudo-Laplacian can be efficiently splitted into a first-order system. Similar results as for elliptic equations are derived. Numerical results for the Helmholtz equations and the spectral Pseudo-Laplacian substantiate the usefulness of our approach.

2 The one-dimensional case

We consider one-dimensional second-order problems given by

$$\begin{aligned} -u'' + \gamma u &= f \ in \ \Omega = (-1,1), && (1)\\ u(-1) = u(1) &= 0. && (2) \end{aligned}$$

Here f denotes the right-hand side and $\gamma \geq 0$ is a given parameter.
The second-order equation is splitted into the first-order system

$$\begin{aligned} u' - v &= 0 \ in \ \overline{\Omega} = [-1,1], && (3)\\ \gamma u - v' &= f \ in \ \Omega = (-1,1), && (4)\\ u(-1) = u(1) &= 0. && (5) \end{aligned}$$

Here v denotes an auxiliary function defined on $[-1,1]$. For the spectral approximation we introduce polynomial subspaces

$$\begin{aligned} \mathbb{P}_N^0 &= \{Polynomials\ of\ degree\ \leq N\ which\ vanish\ in\ \pm 1\},\\ \mathbb{P}_{N-1} &= \{Polynomials\ of\ degree\ \leq N-1\}. \end{aligned}$$

The functions u, v are approximated by polynomials $u_N \in \mathbb{P}_N^0$ and $v_{N-1} \in \mathbb{P}_{N-1}$. Furthermore we have to introduce the standard Gauss- and Gauss-Lobatto-Chebyshev collocation nodes. They are given by

$$\begin{aligned} x_i &= \cos\frac{i\pi}{N},\ i = 0,\dots,N,\\ \hat{x}_i &= \cos\frac{(2i-1)\pi}{2N},\ i = 1,\dots,N. \end{aligned}$$

In the following the function u is defined on the Gauss-Lobatto nodes and v on the Gauss nodes. The pseudo spectral (or collocation) scheme for the first-order system (3-5) now reads as follows: Find functions $u_N \in \mathbb{P}_N^0,\ v_{N-1} \in \mathbb{P}_{N-1}$ such that

$$\begin{aligned} u_N'(\hat{x}_i) - v_{N-1}(\hat{x}_i) &= 0,\ i = 1,\dots,N, && (6)\\ \gamma u_N(x_i) - v_{N-1}'(x_i) &= f(x_i),\ i = 1,\dots,N-1. && (7) \end{aligned}$$

This yields a staggered grid discretization of the first-order system. Since u'_N, v_{N-1} are polynomials of degree $\leq N-1$ and collocation is performed at the N Gauss nodes it is clear that $u'_N \equiv v_{N-1}$. Hence u_N is also identical to the original pseudo spectral solution of the second-order elliptic problem (1-2) on Gauss-Lobatto nodes. The system (6-7) yields $M = 2N-1$ conditions of collocation for the M unknowns $u_N(x_i).i = 1,\ldots,N-1,\ v_{N-1}(\hat{x}_i), i = 1,\ldots,N$. In the following we write the spectral derivatives in matrix notation. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrices:

$$
\begin{aligned}
T_0 &= \left(\cos(k\frac{i\pi}{N})\right),\ i,k = 0,\ldots,N, \\
T_1 &= \left(\cos(k\frac{2i-1\pi}{2N})\right),\ k = 0,\ldots,N-1, i = 1,\ldots,N.
\end{aligned}
$$

Hence we defined matrices $T_0 \in \mathrm{IR}^{N+1,N+1}$ and $T_1 \in \mathrm{IR}^{N,N}$. Further we need the differentation matrix in the Chebyshev coefficient space which is explicitely given by $\hat{D} = (d_{i,j}) \in \mathrm{IR}^{N+1,N+1}$ with

$$
d_{i,j} = \begin{cases} \frac{2j}{c_i}, & j = i+1, i+3, \ldots, N \\ 0, & \text{else} \end{cases}
$$

and

$$
c_i = \begin{cases} 2, & i = 0, \\ 1, & \text{else.} \end{cases}
$$

Now we are able to write the spectral derivative matrices D_0 and D_1 for u and v. They are explicitely given by

$$
D_{0,sp} = T_1 \hat{D} T_0^{-1} \in \mathrm{IR}^{N,N-1}
$$

and

$$
D_{1,sp} = T_0 \hat{D} T_1^{-1} \in \mathrm{IR}^{N-1,N}.
$$

Here in D_0 (resp. D_1) the first and last columns (resp. rows) are omitted. This is due to the homogeneous Dirichlet boundary conditions and the fact that collocation is not performed in boundary points. By further introducing identity matrices $\hat{I} \in \mathrm{IR}^{N-1,N-1}$, $I \in \mathrm{IR}^{N,N}$ we are now able to write the spectral operator $A_{sp} \in \mathrm{IR}^{M,M}$ for the first-order system (6-7):

$$A_{sp} \;=\; \left[\begin{array}{cc} D_{0,sp} & -I \\ \gamma \hat{I} & -D_{1,sp} \end{array} \right].$$

Since A_{sp} only contains first-order derivatives it is clear that its eigenvalues are complex. Furthermore it can be expected that the maximal eigenvalues increase as $O(N^2)$. This was fully confirmed by numerical experiments. We calculated the minimal and maximal eigenvalues λ_{min} and λ_{max} of A_{sp}. More precisely, we define:

$$\begin{aligned} \lambda_{min} &= \min\left\{\mid \lambda \mid :\; \lambda \text{ eigenvalue}\right\}, \\ \lambda_{max} &= \max\left\{\mid \lambda \mid :\; \lambda \text{ eigenvalue}\right\}. \end{aligned}$$

In table I we present these values for $\gamma = 0, 1, 10$ and $N = 4, 8, 16, 32, 64, 128$. Obviously, λ_{max} behaves as $O(N^2)$ for increasing N. Hence the spectral condition number for second-order elliptic equations is reduced from $O(N^4)$ (see [?]) to $O(N^2)$. This yields a desirable improvement with respect to rounding errors. Furthermore our approach has also advantages for preconditioning.

Here we consider finite difference preconditioning. For second-order derivatives it is well known that the preconditioned spectral operator has real eigenvalues (see [?]) confined to the interval $[1, \pi^2/4]$. Haldenwang et al. [?] have analytically calculated the eigenvalues and confirmed this behaviour. An alternative type of preconditioning is based on finite elements (see Deville and Mund [?]). For our first-order splitted system we prefer finite differences. The finite difference matrices corresponding to the spectral derivatives $D_{0,sp}, D_{1,sp}$ are denoted by $D_{0,fd}, D_{1,fd}$. They are explicitely defined by $D_{0,fd} = (d0_{i,j}) \in \mathrm{IR}^{N,N-1}$ where

$$d0_{i,j} \;=\; \left\{ \begin{array}{cl} \frac{1}{x_{i+1}-x_i}\;, & j = i,\; i = 0, \ldots, N-2 \\ -\frac{1}{x_{i+1}-x_i}, & j = i-1,\; i = 1, \ldots, N-1 \\ 0\;, & \text{else} \end{array} \right.$$

and $D_{1,fd} = (d1_{i,j}) \in \mathrm{IR}^{N-1,N}$ where

$$d1_{i,j} \;=\; \left\{ \begin{array}{cl} -\frac{1}{\hat{x}_{i+1}-\hat{x}_i}, & j = i,\; i = 1, \ldots, N-1 \\ \frac{1}{\hat{x}_{i+1}-\hat{x}_i}\;, & j = i+1,\; i = 1, \ldots, N-1 \\ 0\;, & \text{else.} \end{array} \right.$$

Here $D_{0,fd}$ yields the first (finite difference-) derivative of u at the Chebyshev-Gauss nodes evaluated at the Gauss-Lobatto nodes. The matrix $D_{1,fd}$ defines the first derivative of v vice versa. Now we are able to write the finite difference operator $A_{fd} \in \mathbb{R}^{M,M}$ for the first-order system (6-7):

$$A_{fd} \;=\; \begin{bmatrix} D_{0,fd} & -I \\ \gamma \hat{I} & -D_{1,fd} \end{bmatrix}.$$

We were interested in the preconditioning properties and hence calculated the eigenvalues of

$$A_{fd}^{-1} A_{sp}$$

for the same parameters γ and N as before. It was observed that the eigenvalues are real and were confined to the interval $[1, \pi/2]$ (see table II). The eigenvalue one has double multiplicity which is due to the fact that the first derivative occurs twice. The good performance of the preconditioner could also be expected from the paper of Funaro [?] where staggered grid preconditiong was sucessfully employed for the first derivative spectral operator. Hence our staggered grid finite difference preconditioner yields the improved condition number $\pi/2$ instead of $\pi^2/4$. For the (stationary) Richardson relaxation (see [?], [?]) the optimal relaxation parameter ω_{opt} is now given by

$$\omega_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}} = \frac{2}{1 + \pi/2} \cong 0.7780.$$

The corresponding convergence rate ρ is given by

$$\rho = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} = \frac{\pi/2 - 1}{\pi/2 + 1} \cong 0.2220.$$

This is about the half of the usual rate 0.4232 for second-order problems. For the same accuracy only the half number of Richardson iterations is required. Further improvements in convergence speed can be achieved by Chebyshev acceleration (see [?]). For three nonstationary Richardson relaxations we obtain an optimal convergence rate of about 0.1417.

3 The two-dimensional case

We consider Helmholtz equations given by

$$-\Delta u + \gamma u = f \; in \; \Omega = (-1,1)^2, \tag{8}$$

$$u = 0 \; on \; \partial\Omega. \tag{9}$$

Here f denotes the right-hand side and $\gamma \geq 0$ is a given Helmholtz parameter. The second-order equation is splitted into the first-order system

$$u_x - v_1 = 0 \; in \; \overline{\Omega} = [-1,1]^2, \tag{10}$$

$$u_y - v_2 = 0 \; in \; \overline{\Omega} = [-1,1]^2, \tag{11}$$

$$\gamma u - div(v) = f \; in \; \Omega = (-1,1)^2, \tag{12}$$

$$u = 0 \; on \; \partial\Omega. \tag{13}$$

Here $v = (v_1, v_2) \in \mathbb{R}^2$ denotes an auxiliary function defined on $[-1,1]^2$. For the spectral approximation in the 2D-case we introduce polynomial subspaces

$$\begin{aligned} \mathbb{P}^0_{N,N} &= \{Pol.\ of\ degree\ \leq N\ in\ each\ variable\ which\ vanish\ on\ \partial\Omega\}, \\ \mathbb{P}_{N-1,N} &= \{Pol.\ of\ degree\ \leq N-1\ in\ x\ and\ of\ degree\ \leq N\ in\ y\}, \\ \mathbb{P}_{N,N-1} &= \{Pol.\ of\ degree\ \leq N\ in\ x\ and\ of\ degree\ \leq N-1\ in\ y\}. \end{aligned}$$

The function u is approximated by polynomials in $P^0_{N,N}$. v_1 is approximated by polynomials in $\mathbb{P}_{N-1,N}$ and v_2 by polynomials in $\mathbb{P}_{N,N-1}$. As before the function u is defined on the Gauss-Lobatto nodes in both variables x and y. The function v_1 is defined on the Gauss nodes in x and the Gauss-Lobatto nodes in y whereas v_2 is defined on the Gauss-Lobatto nodes in x and the Gauss nodes in y. The pseudo spectral (or collocation) scheme for the first-order system (10-13) now reads as follows: Find functions $u_N \in \mathbb{P}^0_{N,N}$, $v_{1,N} \in \mathbb{P}_{N-1,N}$ and $v_{2,N} \in \mathbb{P}_{N,N-1}$ such that

$$u_{N,x}(\hat{x}_i, y_j) - v_{1,N}(\hat{x}_i, y_j) = 0, \; i = 1, \ldots, N, \; j = 0, \ldots, N \tag{14}$$

$$u_{N,y}(x_i, \hat{y}_j) - v_{2,N}(x_i, \hat{y}_j) = 0, \; i = 0, \ldots, N, \; j = 1, \ldots, N \tag{15}$$

$$\gamma u_N(x_i, y_j) - div(v_N)(x_i, y_j) = f(x_i, y_j), \; i,j = 1, \ldots, N-1. \tag{16}$$

This yields a staggered grid discretization of the first-order system. Due to the degree of the polynomials and the type of collocation it becomes clear that $u_{N,x} \equiv v_{1,N}$ and $u_{N,y} \equiv v_{2,N}$. Hence u_N is also identical to the original pseudo spectral solution of the second-order elliptic problem (8-9) on Gauss-Lobatto nodes. Ths system (14-16) yields $M = 3N^2 + 1$ conditions of collocation for the M unknowns $u_N(x_i, y_j).i, j = 1, \ldots, N-1,\ v_{1,N}(\hat{x}_i, y_j), i = 1, \ldots, N, j = 0, \ldots, N,\ v_{2,N}(x_i, \hat{y}_j), i = 0, \ldots, N, j = 1, \ldots, N$. By tensor product representation

$$A \otimes B = (Ab_{i,j})_{i,j}$$

we are now able to write the 2D discrete spectral operator in matrix notation. For this purpose we first have to introduce the identity matrix $\tilde{I} \in \mathrm{IR}^{N+1,N+1}$ and the matrix $\tilde{I}_0 \in \mathrm{IR}^{N+1,N-1}$ which results from $\tilde{I}$ by omitting the first and last column. Now the spectral operator $B_{sp} \in \mathrm{IR}^{M,M}$ can be written as

$$B_{sp} = \begin{bmatrix} D_{0,sp} \otimes \tilde{I}_0 & -I \otimes \tilde{I} & 0 \\ \tilde{I}_0 \otimes D_{0,sp} & 0 & -\tilde{I} \otimes I \\ \gamma \hat{I} \otimes \hat{I} & -D_{1,sp} \otimes \tilde{I}_0^t & -\tilde{I}_0^t \otimes D_{1,sp} \end{bmatrix}.$$

In table III we present the minimal ad maximal eigenvalues of B_{sp} for $\gamma = 0, 1, 10$ and $N = 4, 8, 16, 32$. Once more λ_{min} approximates $\frac{\pi}{2}$ The maximal eigenvalue is always real and λ_{max} behaves as $O(N^2)$ for increasing N. Hence the spectral condition number for second-order elliptic equations is reduced from $O(N^4)$ to $O(N^2)$. Furthermore our approach has advantages for preconditioning.

The finite difference operator B_{fd} can be generated in the same manner as B_{sp} where the matrices $D_{0,sp}, D_{1,sp}$ have to be replaced by $D_{0,fd}, D_{1,fd}$. Once more we calculated the eigenvalues of the preconditioned spectral operator

$$B_{fd}^{-1} B_{sp}$$

for the same parameters γ and N as before. For $\gamma = 0$ it was observed that the eigenvalues are real and were confined to the interval $[1, \pi/2]$ (see table IV). For increasing γ the eigenvalues become slightly complex with very small imaginary parts. Hence our staggered grid finite difference preconditioner yields the improved condition number $\pi/2$ instead of $\pi^2/4$. For the (stationary) Richardson relaxation the optimal convergence rate ρ is given by

$$\rho = \frac{\pi/2 - 1}{\pi/2 + 1} \cong 0.2220.$$

A similar discussion as in the 1D-case can be performed. For the efficient solution of the first-order finite difference preconditioner we propose a least-squares solver (see [?],[?]). Here we recommend multigrid cycles where line Gauss-Seidel or incomplete LU (ILU) decompositions are used for relaxation. For more details we refer to the earlier papers of Heinrichs [?],[?] or Zang et al. [?],[?].

4 The spectral Pseudo-Laplacian

4.1 One-dimensional case

By a splitting technique (see [?]) the Navier-Stokes equations can be decoupled into Helmholtz equations for the velocity components and an equation with the Pseudo-Laplacian for the pressure. We are already able to handle Helmholtz equations. Now we consider the one-dimensional Pseudo-Laplacian. Here we employ the staggered grid discretization introduced in [?]. The pressure p is defined on the Chebyshev-Gauss nodes and we introduce an intermediate function q which is defined on the Gauss-Lobatto nodes and vanishes in ± 1. The pseudo spectral (or collocation) scheme for the Pseudo-Laplacian now reads as follows: Find functions $p_{N-1} \in \mathbb{P}_{N-1},\ q_N \in \mathbb{P}_N^0$ such that

$$
\begin{aligned}
p'_{N-1}(x_i) - q_N(x_i) &= 0,\ i = 1, \ldots, N-1, & (17)\\
q'_N(\hat{x}_i) &= g(\hat{x}_i),\ i = 1, \ldots, N. & (18)
\end{aligned}
$$

This yields a first-order staggered grid discretization of the Pseudo-Laplacian. The system (17-18) yields $M = 2N - 1$ conditions of collocation for the Munknowns $p_{N-1}(\hat{x}_i).i = 1, \ldots, N,\ q_N(x_i), i = 1, \ldots, N-1$. The spectral derivative matrices D_0 and D_1 for p and q are explicitely given by

$$D_{0,sp} = T_0 \hat{D} T_1^{-1} \in \mathbb{R}^{N-1,N}$$

and

$$D_{1,sp} = T_1 \hat{D} T_0^{-1} \in \mathbb{R}^{N,N-1}.$$

Here in D_0 (resp. D_1) the first and last rows (resp. columns) are omitted. This is due to the homogeneous Dirichlet boundary conditions and the fact that collocation is not performed in boundary points. We are now able to write the spectral operator $C_{sp} \in \mathrm{IR}^{M,M}$ for the first-order system (17-18):

$$C_{sp} \;=\; \left[\begin{array}{cc} D_{0,sp} & -\hat{I} \\ 0 & D_{1,sp} \end{array}\right].$$

Since C_{sp} only contains first-order derivatives it is clear that its eigenvalues are complex. One eigenvalue is zero due to the fact that the pressure is only determined up to a constant. Furthermore it can be expected that the maximal eigenvalues increase as $O(N^2)$. This was fully confirmed by numerical experiments (see table V). Furthermore the staggered grid approach has advantages for preconditioning.

Here we once more consider finite difference preconditioning. The finite difference matrices corresponding to the spectral derivatives $D_{0,sp}, D_{1,sp}$ are denoted by $D_{0,fd}, D_{1,fd}$. They are explicitely defined by $D_{0,fd} = (d0_{i,j}) \in \mathrm{IR}^{N-1,N}$ where

$$d0_{i,j} \;=\; \left\{\begin{array}{cc} -\frac{1}{\hat{x}_{i+1}-\hat{x}_i}, & j=i,\; i=1,\ldots,N-1 \\ \frac{1}{\hat{x}_{i+1}-\hat{x}_i}, & j=i+1,\; i=1,\ldots,N-1 \\ 0\quad, & \text{else} \end{array}\right.$$

and $D_{1,fd} = (d1_{i,j}) \in \mathrm{IR}^{N,N-1}$ where

$$d1_{i,j} \;=\; \left\{\begin{array}{cc} \frac{1}{x_i-x_{i-1}}\;, & j=i,\; i=1,\ldots,N \\ -\frac{1}{x_i-x_{i-1}}, & j=i-1,\; i=1,\ldots,N \\ 0\quad, & \text{else.} \end{array}\right.$$

Here $D_{0,fd}$ yields the first (finite difference-) derivative of p at the Chebyshev-Gauss-Lobatto nodes evaluated at the Gauss nodes. The matrix $D_{1,fd}$ defines the first derivative of q vice versa. Now we are able to write the finite difference operator $C_{fd} \in \mathrm{IR}^{M,M}$ for the Pseudo-Laplacian:

$$C_{fd} \;=\; \left[\begin{array}{cc} D_{0,fd} & -\hat{I} \\ 0 & D_{1,fd} \end{array}\right].$$

Once more C_{fd} is a singular matrix. Since we are interested in preconditioning properties we have to regularize both operators C_{sp} and C_{fd}. By eliminating the Nth column and Mth row we obtain nonsingular matrices from $\mathbb{R}^{M-1,M-1}$. They are denoted by $\hat{C}_{sp}$ and $\hat{C}_{fd}$. Then we calculated the eigenvalues of

$$\hat{C}_{fd}^{-1}\hat{C}_{sp}.$$

Once more it was observed that the eigenvalues are real and were confined to the interval $[1, \pi/2]$ (see table V). Hence our staggered grid finite difference preconditioner yields the improved condition number $\pi/2$ instead of $\pi^2/4$. This once more leads to an improved convergence speed of the Richardson relaxation. The same discussion as in section 2 can be performed.

4.2 The two-dimensional case

For the spectral approximation of the two-dimensional Pseudo-Laplacian we introduce polynomial subspaces

$$\begin{aligned}
\mathbb{P}^0_{N,N-1} &= \{Pol.\ of\ deg.\ \leq N\ in\ x\ and\ \leq N-1\ in\ y\ vanishing\ in\ x=\pm 1\} \\
\mathbb{P}^0_{N-1,N} &= \{Pol.\ of\ deg.\ \leq N-1\ in\ x\ and\ \leq N\ in\ y\ vanishing\ in\ y=\pm 1\}
\end{aligned}$$

and

$$\mathbb{P}_{N-1,N-1} = \{Polynomials\ of\ degree\ \leq N-1\ in\ x\ and\ y\}.$$

The pressure p is approximated by polynomials in $P_{N-1,N-1}$. Intermediate functions $q_1 \in \mathbb{P}^0_{N,N-1}$ and $q_2 \in \mathbb{P}^0_{N-1,N}$ are introduced. As before the pressure p is defined on the Gauss nodes in both variables x and y. The function q_1 is defined on the Gauss-Lobatto nodes in x and the Gauss nodes in y whereas q_2 is defined on the Gauss nodes in x and on the Gauss-Lobatto nodes in y. The pseudo spectral (or collocation) scheme for the first-order splitting of the Pseudo-Laplacian now reads as follows: Find functions $p_{N-1} \in \mathbb{P}_{N-1,N-1}$, $q_{1,N} \in \mathbb{P}^0_{N,N-1}$ and $q_{2,N} \in \mathbb{P}^0_{N-1,N}$ such that

$$\begin{aligned}
p_{N-1,x}(x_i, \hat{y}_j) - q_{1,N}(x_i, \hat{y}_j) &= 0,\ i = 1, \ldots, N-1,\ j = 1, \ldots, N & (19) \\
p_{N-1,y}(\hat{x}_i, y_j) - q_{2,N}(\hat{x}_i, y_j) &= 0,\ i = 1, \ldots, N,\ j = 1, \ldots, N-1 & (20) \\
div(q_N)(\hat{x}_i, \hat{y}_j) &= g(\hat{x}_i, \hat{y}_j),\ i, j = 1, \ldots, N. & (21)
\end{aligned}$$

This yields a staggered grid discretization of the first-order system. The system (19-21) yields $M = 2N(N-1) + N^2 = 3N^2 - 2N$ conditions of collocation for the M unknowns. By tensor product representation we are once more able to write the 2D discrete spectral operator in matrix notation. Now the spectral operator $D_{sp} \in \mathrm{I\!R}^{M,M}$ can be written as follows:

$$D_{sp} \;=\; \left[\begin{array}{ccc} D_{0,sp} \otimes I & -\hat{I} \otimes I & 0 \\ I \otimes D_{0,sp} & 0 & -I \otimes \hat{I} \\ 0 & D_{1,sp} \otimes I & I \otimes D_{1,sp} \end{array} \right].$$

From table VI it can be observed that once more λ_{max} behaves as $O(N^2)$ for increasing N. One eigenvalue is zero due to the fact that the pressure is not uniquely determined. Furthermore our approach has advantages for preconditioning. The finite difference operator D_{fd} can be generated in the same manner as D_{sp} where the matrices $D_{0,sp}, D_{1,sp}$ have to be replaced by $D_{0,fd}, D_{1,fd}$. We obtained regular operators by eliminating the $N^2 th$ column and the Mth row. The corresponding matrices from $\mathrm{I\!R}^{M-1,M-1}$ are denoted by $\hat{D}_{sp}$ and $\hat{D}_{fd}$. Once more we calculated the eigenvalues of the preconditioned spectral operator

$$\hat{D}_{fd}^{-1}\hat{D}_{sp}.$$

The eigenvalues become slightly complex with very small imaginary parts. The maximal eigenvalue is still approximating $\pi/2$ (see table VI). Hence also for the Pseudo-Laplacian we obtain improved preconditioning properties by splitting it into a first-order system. For the efficient solution of the first-order finite difference preconditioner we once more recommend a least-squares solver based on line Gauss-Seidel relaxations or ILU decompositions.

Conclusion

A least squares solver for the pseudo spectral (or collocation) approximation of elliptic equations is presented. The elliptic spectral system is decoupled into a first-order system. The results are extended to the spectral Pseudo-Laplacian. A staggered grid spectral discretization is introduced. By suitable finite difference preconditioning the condition number is reduced to $\frac{\pi}{2}$. Efficient least-squares solvers based on multigrid with line relaxation are proposed. Numerical results are presented which substantiate the usefulness of our concept.

TABLES

	$\gamma = 0$		$\gamma = 1$		$\gamma = 10$	
N	λ_{min}	λ_{max}	λ_{min}	λ_{max}	λ_{min}	λ_{max}
4	2.12	$4.87 \cdot 10^0$	1.73	$4.85 \cdot 10^0$	2.96	$4.82 \cdot 10^0$
8	1.67	$1.99 \cdot 10^1$	1.96	$1.99 \cdot 10^1$	3.51	$1.99 \cdot 10^1$
16	1.61	$7.85 \cdot 10^1$	1.91	$7.85 \cdot 10^1$	3.70	$7.85 \cdot 10^1$
32	1.58	$3.13 \cdot 10^2$	1.88	$3.13 \cdot 10^2$	3.59	$3.13 \cdot 10^2$
64	1.57	$1.25 \cdot 10^3$	1.87	$1.25 \cdot 10^3$	3.55	$1.25 \cdot 10^3$
128	1.57	$5.00 \cdot 10^3$	1.86	$5.00 \cdot 10^3$	3.54	$5.00 \cdot 10^3$

Table I. λ_{min} and λ_{max} of A_{sp}

	$\gamma = 0$		$\gamma = 1$		$\gamma = 10$	
N	λ_{min}	λ_{max}	λ_{min}	λ_{max}	λ_{min}	λ_{max}
4	1.00	1.5307	1.00	1.4526	1.00	1.2047
8	1.00	1.5607	1.00	1.5232	1.00	1.3738
16	1.00	1.5683	1.00	1.5493	1.00	1.4761
32	1.00	1.5702	1.00	1.5607	1.00	1.5276
64	1.00	1.5706	1.00	1.5659	1.00	1.5506
128	1.00	1.5708	1.00	1.5684	1.00	1.5611

Table II. λ_{min} and λ_{max} of $A_{fd}^{-1} A_{sp}$

	$\gamma = 0$		$\gamma = 1$		$\gamma = 10$	
N	λ_{min}	λ_{max}	λ_{min}	λ_{max}	λ_{min}	λ_{max}
4	1.21	$3.19{\cdot}10^0$	1.29	$3.19{\cdot}10^0$	1.42	$3.19{\cdot}10^0$
8	1.40	$7.84{\cdot}10^0$	1.74	$7.84{\cdot}10^0$	2.00	$7.84{\cdot}10^0$
16	2.17	$2.57{\cdot}10^1$	2.18	$2.57{\cdot}10^1$	2.26	$2.57{\cdot}10^1$
32	2.06	$1.16{\cdot}10^2$	2.14	$1.16{\cdot}10^2$	2.55	$1.16{\cdot}10^2$

Table III. λ_{min} and λ_{max} of B_{sp}

	$\gamma = 0$		$\gamma = 1$		$\gamma = 10$	
N	λ_{min}	λ_{max}	λ_{min}	λ_{max}	λ_{min}	λ_{max}
4	1.00	1.5307	1.00	1.4918	1.00	1.2644
8	1.00	1.5607	1.00	1.5424	1.00	1.4145
16	1.00	1.5683	1.00	1.5593	1.00	1.4965
32	1.00	1.5702	1.00	1.5658	1.00	1.5357

Table IV. λ_{min} and λ_{max} of $B_{fd}^{-1} B_{sp}$

	$\hat{C}_{sp}$		$\hat{C}_{fd}^{-1}\hat{C}_{sp}$	
N	λ_{min}	λ_{max}	λ_{min}	λ_{max}
4	1.14	$5.40{\cdot}10^0$	1.00	1.5307
8	2.07	$9.25{\cdot}10^0$	1.00	1.5607
16	2.98	$3.54{\cdot}10^1$	1.00	1.5683
32	3.89	$1.30{\cdot}10^2$	1.00	1.5702
64	4.77	$5.60{\cdot}10^2$	1.00	1.5706
128	5.60	$2.24{\cdot}10^3$	1.00	1.5708

Table V. λ_{min} and λ_{max} of $\hat{C}_{sp}$ and $\hat{C}_{fd}^{-1}\hat{C}_{sp}$

	$\hat{D}_{sp}$		$\hat{D}_{fd}^{-1}\hat{D}_{sp}$	
N	λ_{min}	λ_{max}	λ_{min}	λ_{max}
4	0.78	$5.45{\cdot}10^0$	1.00	1.5307
8	0.91	$1.84{\cdot}10^1$	1.00	1.5607
16	0.99	$7.45{\cdot}10^1$	1.00	1.5683
32	1.02	$3.04{\cdot}10^2$	1.00	1.5702

Table VI. λ_{min} and λ_{max} of $\hat{D}_{sp}$ and $\hat{D}_{fd}^{-1}\hat{D}_{sp}$

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