OPERATOR SPLITTING
FOR THE
STOKES EQUATIONS

by

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Abstract

A pseudo spectral (or collocation) approximation of the unsteady Stokes equations is presented. Using the Uzawa algorithm the spectral system is decoupled into Helmholtz equations for the velocity components and an equation with the Pseudo-Laplacian for the pressure. In order to avoid spurious modes the pressure is approximated with lower order (two degrees lower) polynomials than the velocity. Only one grid (no staggered grids) with the standard Chebyshev Gauss-Lobatto nodes is used. Here we further compare our treatment with a Neumann boundary value problem for the pressure. The highly improved accuracy of our method becomes obvious. In the time discretization a high order backward differentation scheme for the intermediate velocity is combined with a high order extrapolant for the pressure. It is numerically shown that a stable third order method in time can be achieved. Higher order schemes can be obtained by a direct Uzawa decoupling without pressure correction. Numerical results are presented.

1 Introduction

We consider the unsteady Stokes equations in velocity-pressure formulation which are discretized by a pseudo spectral (or collocation) method. This means that the solution is approximated by global Chebyshev polynomials (see, e.g., Canuto et al. [7] and Orszag [22]). The collocation nodes are the standard Chebyshev Gauss-Lobatto nodes. In particular, we introduce a Chebyshev collocation method which has the following desirable properties:

- high spectral accuracy
- no spurious modes except the physical one (constant mode)
- nonstaggered grids.

If the Stokes equations are spectrally discretized in a straightforward manner, the pressure is affected by spurious modes (parassite modes) which deteriorate the spectral accuracy of the method. This is usually avoided by introducing staggered grids where both the Gauss nodes and the Gauss-Lobatto nodes are employed (see, e.g., Bernardi and Maday [4]). This problem is also known from finite difference (checkerboard instability) or finite element (Babuška – Brezzi condition) discretizations. Here we avoid the occurence of spurious modes by approximating the pressure with lower order polynomials than the velocity. The same idea was already mentioned by Schumack, Schultz and Boyd [26] in a similar context. In [26] numerous ways to avoid the problem of spurious modes are presented. The pseudo spectral methods using nonstaggered grids are simpler to implement and have comparable or better accuracy than the staggered grid formulation. A convergence analysis of this method was already given by Azaiez [2], [3].

Our approach to solve the spectral system is to use a global iterative decoupling procedure which is an extension of the classical Uzawa algorithm. The Uzawa algorithm was already extensivly used in the finite element context (see Arrow, Hurwitz and Uzawa [1], Bristeau, Glowinski and Periaux [6], Girault and Raviart [8], Glowinski [9], Temam [27]). This approach was also chosen by Rønquist [25] in the context of spectral element methods. The Uzawa method is more efficient in terms of computational complexity than a direct approach (see Yamaguchi, Cheng and Brown [28]). In particular, this becomes obvious for large three-dimensional systems. It requires no pressure boundary conditions or rediscretizations of the original problem. In so far it is more attractive than the Poisson equation approach (Glowinski and Pironneau [10], Kleiser and Schumann [20]). A quite good survey about operator splitting techniques is also given by Maday et al. [21].

We present a decoupling procedure where we reduce the problem to the solution of Helmholtz equations and an equation with a "*Pseudo–Laplace*" operator (also called "*energy*"). This splitting was introduced in [18] where the properties of this operator were studied. It was shown that the eigenvalues of the spectral Pseudo-Laplacian are real and negative (except one eigenvalue which is zero and belongs to the constant mode). By using this result it is readily seen that our discretization does not introduce any spurious modes for the pressure. Since the spectral Pseudo-Laplacian is very ill-conditioned an effective iterative solver requires a preconditioner. Hence in [18] suitable finite difference (FD) preconditioners are presented. These results are important since the Pseudo–Laplacian occurs in many applications. For instance also the spectral multigrid (SMG) Stokes solver by *distributive relaxations* (see Heinrichs [17]) makes use of these results. Performing a distributive relaxation only involves smoothing Poisson–like equations. Hereby the Pseudo–Laplace operator occurs.

Here we further compare our treatment with a Neumann boundary value problem for the pressure (see Rannnacher [24]). The highly improved accuracy of our method becomes obvious. The Neumann boundary condition yields no correct condition for the pressure where no boundary conditions are prescribed. In the time discretization a high order backward differentation scheme for the intermediate velocity is combined with a high order extrapolant for the pressure. It is numerically shown that a stable third order method in time can be achieved. Unfortunately, the corresponding fourth order method becomes unstable. Hence we recommend our third order method for sufficiently small time steps. Higher order schemes can be obtained by a direct Uzawa decoupling without pressure correction. The resulting scheme is one order less accurate (since there is no pressure correction) but it remains to be stable up to an order of six. Numerical results are presented.

2 Time Splitting scheme

We consider the unsteady Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla^2 u + \nabla p &= f \;\text{ in } \Omega = (-1,1)^2,\; t>0, && (1)\\ \nabla\cdot u &= 0 \;\text{ in } \Omega,\; t>0, && (2)\\ u &= 0 \;\text{ on } \partial\Omega,\; t>0, && (3) \end{aligned}$$

where $u = (u_1, u_2)^t$ denotes the velocity and p the pressure. f is a given force defined in Ω. The problem is subject to an initial condition at $t = 0$. Since the pressure is only determined up to a constant, we impose the average pressure to be zero, i.e., $\int_\Omega p dx = 0$.
The standard backward Euler time discretization of the equations (1)–(3) leads to the following scheme:

$$
\begin{aligned}
L^n_{t,k} u^{n+1} \;-\; \nabla^2 u^{n+1} \;+\; \nabla p^{n+1} &= f^{n+1} \;\text{ in } \Omega, &(4)\\
\nabla \cdot u^{n+1} &= 0 \;\text{ in } \Omega, &(5)\\
u^{n+1} &= 0 \;\text{ on } \partial\Omega, &(6)
\end{aligned}
$$

where Δt denotes the step size in t and the index $n+1$ indicates that the functions are evaluated at the time step $t_{n+1} = (n+1)\Delta t$ for $n = 0, 1, 2, \ldots$. The operator $L^n_{t,k}$ represents the backward differentation scheme for the approximation of $\frac{\partial}{\partial t}$. The index k denotes the order of the scheme. It can be written as:

$$
L^n_{t,k} u^{n+1} = \frac{1}{\Delta t} \sum_{m=0}^{k} \beta_m u^{n+1-m} \tag{7}
$$

For $k = 1$ the operator $L^n_{t,1}$ yields the standard backward Euler scheme given by $\beta_0 = 1,\ \beta_1 = -1$:

$$
L^n_{t,1} u^{n+1} = \frac{u^{n+1} - u^n}{\Delta t}
$$

which is of first order in time. For $k = 2$ we obtain $\beta_0 = \frac{3}{2},\ \beta_1 = -2,\ \beta_2 = \frac{1}{2}$:

$$
L^n_{t,2} u^{n+1} = \frac{\frac{3}{2} u^{n+1} - 2u^n + \frac{1}{2} u^{n-1}}{\Delta t}
$$

which is of second order accuracy in the approximation of $\frac{\partial}{\partial t}$.

In order to minimize the computational cost it is reasonable to introduce an operator splitting approach, in which each of the spatial operators (here for the velocity and the pressure) is treated explicitly. Here we consider the following splitting:

$$
\begin{aligned}
L^n_{t,k} \tilde{u}^{n+1} \;-\; \nabla^2 \tilde{u}^{n+1} \;+\; \nabla \overline{p}^{n+1}_l &= f^{n+1} \;\text{ in } \Omega, &(8)\\
\tilde{u}^{n+1} &= 0 \;\text{ on } \partial\Omega &(9)
\end{aligned}
$$

and

$$\begin{aligned}
\beta_0 \frac{u^{n+1} - \tilde{u}^{n+1}}{\Delta t} \; + \; \nabla \left(p^{n+1} - \overline{p}_l^{n+1} \right) &= 0 \;\text{ in } \Omega, && (10)\\
\nabla \cdot u^{n+1} &= 0 \;\text{ in } \Omega, && (11)\\
u^{n+1} &= 0 \;\text{ on } \partial\Omega. && (12)
\end{aligned}$$

Here $\tilde{u}^{n+1}$ denotes some intermediate velocity and $\overline{p}_l^{n+1}$ defines some extrapolant for the pressure obtained from previous l time steps. Hence at each time step, an intermediate velocity field $\tilde{u}^{n+1}$ is first computed implicitly by using the extrapolated value $\overline{p}_l^{n+1}$ for the pressure, without enforcing the incompressibility constraint. Then, a correction $u^{n+1} - \tilde{u}^{n+1}$ is calculated, as well as a correction to the extrapolated pressure field, $p^{n+1} - \overline{p}_l^{n+1}$. Here the incompressiblity condition is enforced at time t_{n+1}. We are interested in the stability and the order of convergence of the above splitted scheme. Clearly, the order of convergence must depend on the order k of the backward differentation scheme and the order l of the extrapolation where

$$\overline{p}_l^{n+1} \;=\; \sum_{m=0}^{l-1} \gamma_m p^{n-m}.$$

The values $\gamma_m,\ m = 0, \ldots, l-1$ denote suitable extrapolation coefficients. For $l = 1$ we have $\gamma_0 = 1$, for $l = 2$ we have $\gamma_0 = 2,\ \gamma_1 = -1$. The first system (8)-(9) is essentially equivalent to

$$\begin{aligned}
(-\nabla^2 \; + \; \beta_0 \frac{1}{\Delta t} I)\tilde{u}^{n+1} &= \tilde{f}^{n+1} \;\text{ in } \Omega, && (13)\\
\tilde{u}^{n+1} &= 0 \;\text{ on } \partial\Omega && (14)
\end{aligned}$$

where

$$\tilde{f}^{n+1} \;=\; f^{n+1} \; - \frac{1}{\Delta t} \sum_{m=1}^{k} \beta_m u^{n+1-m} - \nabla \overline{p}_l^{n+1}.$$

For the Helmholtz problem (13)–(14) many effective solvers are already available. In particular, we recommend the use of spectral multigrid (SMG) techniques (see [5], [12], [13], [14], [15] , [16], [29], [30]). Above all, the algebraic SMG methods (see [15]) yield very efficient solvers for this type of equation. The system (10)–(12) is equivalent to

$$\beta_0 \frac{1}{\Delta t} u^{n+1} \; + \; \nabla p^{n+1} \;\; = \;\; g^{n+1} \;\text{ in }\; \Omega, \qquad (15)$$

$$\nabla \cdot u^{n+1} = 0 \text{ in } \Omega, \tag{16}$$
$$u^{n+1} = 0 \text{ on } \partial\Omega, \tag{17}$$

where $g^{n+1} = \beta_0 \frac{1}{\Delta t} \tilde{u}^{n+1} + \nabla \overline{p}_l^{n+1}$. For the effective solution we recommend an Uzawa-like (see [1]) decoupling procedure. But before we describe the Uzawa algorithm let us first introduce the pseudo spectral discretization of the above system.

3 Pseudo spectral discretization

In order to give the pseudo spectral discretization of the Stokes systems we have to define spectral derivative operators for the velocity and the pressure. We first introduce these operators in the one-dimensional case. Then the two-dimensional spectral operators can easily be derived by using tensor product representation.

We approximate the velocity components $u_1,\ u_2$ by polynomials in $\mathbb{P}_N^0$, $N \in 2\mathbb{N}$ and the pressure p by polynomials in $\mathbb{P}_{N-2}$. A similar treatment is possible for odd N. Here we use the pseudo spectral Chebyshev discretization. The corresponding spectral derivative operator for the velocity components is for instance given in [7]. Let v denote a scalar function with $v(\pm 1) = 0$. After eliminating the boundary conditions we obtain

$$v'(x_j) \cong (D^0 v)_j, \quad j = 1, \ldots, N-1,$$

where $D^0 \in \mathbb{R}^{N-1,N-1}$, $x_j = \cos \frac{j\pi}{N}$, $j = 1, \ldots, N-1$. Here $D^0 v$ means that D^0 is multiplied by the vector with the components $v(x_j)$, $j = 1, \ldots, N-1$. After eliminating the homogeneous Dirichlet boundary conditions we obtain for the second derivative:

$$v''(x_j) \cong ((D^2)^0 v)_j, \quad j = 1, \ldots, N-1,$$

where $(D^2)^0 \in \mathbb{R}^{N-1,N-1}$.

In order to avoid spurious modes in pressure (see, e.g., Canuto et al. [7]) we approximate the pressure by polynomials in $\mathbb{P}_{N-2}$. In [4] and [25] the discretization for the pressure was performed on a staggered grid induced by the Chebyshev Gauss nodes. The Gauss nodes were mainly chosen for reasons of accomplishing an analysis of the spectral method. A disadvantage of the Gauss nodes lies in the fact that one has to interpolate between the Gauss-Lobatto and Gauss nodes. Hence we prefer nonstaggered grids with the Gauss-Lobatto nodes only. Here we observed no spurious modes, either.

The derivative operator D for the pressure is now defined as follows:

$$p'(x_j) \;\cong\; (Dp)_j, \quad j = 1, \ldots, N-1,$$

where $D \in \mathbb{R}^{N-1,N-1}$ is given by

$$D \;=\; T\hat{D}T^{-1}$$

and $T = (t_{i,k})_{i,k=1,\ldots,N-1}$ is explicitly defined as:

$$t_{i,k} \;:=\; \cos(k-1)\frac{i\pi}{N}, \quad i,k = 1, \ldots, N-1.$$

$\hat{D}$ defines the derivative operator in coefficient space (see [11]). This means that T^{-1} performs the interpolation into $\mathbb{P}_{N-2}$ which is induced by the Chebyshev Gauss-Lobatto nodes $x_j,\ j = 1, \ldots, N-1$. $\hat{D}$ performs the differentiation in coefficient space and T transforms back into physical space.

In the two-dimensional case the spectral derivative operator can easily be constructed using the tensor product representation. Now the partial derivative operators for the velocity read as follows:

$$\begin{aligned}
\frac{\partial}{\partial x} \quad &: \quad D_x^0 \;:=\; D^0 \otimes I, \quad \frac{\partial}{\partial y}: \quad D_y^0 \;:=\; I \otimes D^0, \\
\frac{\partial^2}{\partial x^2} \quad &: \quad D_{xx}^0 \;:=\; (D^2)^0 \otimes I, \quad \frac{\partial^2}{\partial y^2}: \quad D_{yy}^0 \;:=\; I \otimes (D^2)^0, \\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad &: \quad D_\Delta^0 \;:=\; D_{xx}^0 + D_{yy}^0,
\end{aligned}$$

where $I \in \mathbb{R}^{N-1,N-1}$ denotes the identity matrix.
For the pressure we define:

$$\frac{\partial}{\partial x}: \quad D_x \;:=\; D \otimes I, \quad \frac{\partial}{\partial y}: \quad D_y \;:=\; I \otimes D.$$

Now we are ready to present the pseudo spectral discretization of the Stokes system. We first consider the system (4)–(6):

$$\begin{aligned}
\left(-D^0_\Delta + \beta_0 \frac{1}{\Delta t} I\right) u_1^{n+1} \; + \; D_x p^{n+1} \quad &= \quad \hat{f}_1^{n+1}, \qquad (18)\\
\left(-D^0_\Delta + \beta_0 \frac{1}{\Delta t} I\right) u_2^{n+1} \; + \; D_y p^{n+1} \quad &= \quad \hat{f}_2^{n+1}, \qquad (19)\\
D^0_x u_1^{n+1} \; + \; D^0_y u_2^{n+1} \quad &= \quad 0,\\
u_1^{n+1} \; = \; u_2^{n+1} \quad &= \quad 0 \;\; \text{on } \partial\Omega,
\end{aligned}$$

where

$$\hat{f}_i^{n+1} \; := \; f_i^{n+1} \; - \; \frac{1}{\Delta t} \sum_{m=1}^{k} \beta_m u_i^{n+1-m}, \; i = 1, 2.$$

Since we want to use the operator splitting we have to solve a Helmholtz problem for the intermediate velocity components:

$$\begin{aligned}
\left(-D^0_\Delta + \beta_0 \frac{1}{\Delta t}\right) \tilde{u}_i^{n+1} \quad &= \quad \tilde{f}_i^{n+1},\\
\tilde{u}_i^{n+1} \quad &= \quad 0 \;\; \text{on } \partial\Omega, \quad i = 1, 2
\end{aligned}$$

and

$$\begin{aligned}
\beta_0 \frac{1}{\Delta t} u_1^{n+1} + D_x p^{n+1} \quad &= \quad g_1^{n+1}\\
\beta_0 \frac{1}{\Delta t} u_2^{n+1} + D_y p^{n+1} \quad &= \quad g_2^{n+1}\\
D^0_x u_1^{n+1} + D^0_y u_2^{n+1} \quad &= \quad 0 \;\; \text{in } \Omega,\\
u_1^{n+1} = u_2^{n+1} \quad &= \quad 0 \;\; \text{on } \partial\Omega.
\end{aligned}$$

By applying the divergence to the first two equations and further using the divergence - free condition (Uzawa decoupling) we finally arrive at:

$$\left(D^0_x D_x + D^0_y D_y\right) p^{n+1} = D^0_x g_1^{n+1} + D^0_y g_2^{n+1}. \qquad (20)$$

In many applications the operator

$$B = D^0_x D_x + D^0_y D_y$$

is also called *Pseudo-Laplacian* or *energy*. The Helmholtz problems can efficiently be solved by using spectral multigrid techniques as already proposed in [5], [12], [13], [14], [15], [16], [29], [30]. In [18] we already studied the properties of B. It was shown that the eigenvalues of B are real and negative (except one eigenvalue which is zero and belongs to the constant mode). Hence it is readily seen that our discretization does not introduce any spurious modes. Since the spectral Pseudo-Laplacian is very ill-conditioned an effective iterative solver requires a preconditioner. Therefore in [18] suitable finite difference preconditioners are presented. For further details about the Pseudo-Laplacian we refer to [18]. Here we are mainly interested in the accuracy of our splitted scheme. In particular, we are interested in a compare of the above discretization with the Pseudo-Laplacian and a discretization where the pressure is determined due to Neumann boundary conditions (see Rannacher [24]). Here the function p^{n+1} is determined by the following Neumann boundary value problem:

$$\Delta p^{n+1} = \nabla \cdot g^{n+1} \text{ in } \Omega, \tag{21}$$

$$\frac{\partial}{\partial \nu} p^{n+1} = 0 \text{ on } \partial\Omega, \tag{22}$$

where ν denotes the outer unit normal on $\partial\Omega$. These boundary conditions are chosen since

$$\nu \cdot u^{n+1} = \nu \cdot \tilde{u}^{n+1} = 0 \quad \text{on } \partial\Omega.$$

On the other hand it is well known that the exact pressure p satisfies a nonhomogeneous Neumann boundary value problem. Nevertheless the method represents correctly the velocity field. The finite element analysis of Rannacher [24] indicates that in the interior of the domain Ω the pressure p^{n+1} is indeed a reasonable approximation of the exact pressure $p(t_{n+1})$, as the effects of the non-physical Neumann boundary conditions decay exponentially with respect to $dist(x, \partial\Omega)/\sqrt{\Delta t}$, and hence lives only in a narrow boundary layer of width $O(\sqrt{\Delta t})$.

4 The steady case

Here we numerically compare our Pseudo-Laplace equation (20) with the Neumann boundary value problem (21-22). In particular, we consider the following example where

$$\overline{x} = \frac{1}{2}(x+1),\ \overline{y} = \frac{1}{2}(y+1)$$

are variables defined in $[0,1]$ and

$$\begin{aligned} u_1(x,y) &= r(\overline{x})s(\overline{y}),\ u_2(x,y) = -s(\overline{x})r(\overline{y}), && (23)\\ p(x,y) &= exp(\overline{x}+\overline{y}-1), && (24) \end{aligned}$$

where

$$\begin{aligned} r(\overline{x}) &= \overline{x}^2(1-\overline{x})^2\\ s(\overline{y}) &= 2\overline{y}(1-\overline{y})^2 - 2\overline{y}^2(1-\overline{y}). \end{aligned}$$

Hence the functions $u_1,\ u_2$ satisfy

$$u_1(x,y) = u_2(x,y) = 0 \ \text{ for } (x,y) \in \partial\Omega.$$

Furthermore the functions r, s are chosen such that $r' = s$. Hence it follows

$$\begin{aligned} \frac{\partial}{\partial x}u_1(x,y) + \frac{\partial}{\partial y}u_2(x,y) &= \frac{1}{2}\left(r'(\overline{x})s(\overline{y}) - s(\overline{x})r'(\overline{y})\right)\\ &= \frac{1}{2}\left(s(\overline{x})s(\overline{y}) - s(\overline{x})s(\overline{y})\right) = 0. \end{aligned}$$

Therefore the velocity field is divergence-free. For sufficiently small Δt we calculated the pseudo spectral approximation due to the equation with the Pseudo-Laplacian. These approximations were denoted by v_1^N, v_2^N, q^N. They were compared with the solution of the Neumann boundary value problem denoted by $\hat{v}_1^N, \hat{v}_2^N, \hat{q}^N$. We calculated the absolute errors:

$$ERU1 = ||u_1 - v_1^N||_2,\ ERU2 = ||u_2 - v_2^N||_2,\ ERP = ||p - q^N||_2$$

for (20) and

$$ERU1 = ||u_1 - \hat{v}_1^N||_2,\ ERU2 = ||u_2 - \hat{v}_2^N||_2,\ ERP = ||p - \hat{q}^N||_2$$

for (21-22). Here $||\cdot||_2$ denotes the discrete L^2-norm in Ω, i.e.,

$$||v||_2 = \left[\sum_{i,j=0}^{N} v(x_i, y_j)^2\right]^{\frac{1}{2}} /N.$$

We observed that the distribution of the error was uniform in Ω. There were no boundary layers in the pressure. Hence the L^2-norm yields a reasonable measure for the error.

We calculated the above errors by taking sufficient time steps until convergence occured. The calculations were performed for $N = 8$ and $N = 12$. The corresponding numerical results are presented in table I for (22) and table II for (20).

N	$ERU1 = ERU2$	ERP
8	$5.459 \cdot 10^{-4}$	$2.726 \cdot 10^{-2}$
12	$1.131 \cdot 10^{-4}$	$9.718 \cdot 10^{-3}$

Table I. Results for the Neumann condition (22)

N	$ERU1 = ERU2$	ERP
8	$1.930 \cdot 10^{-9}$	$8.456 \cdot 10^{-8}$
12	$6.751 \cdot 10^{-16}$	$4.770 \cdot 10^{-14}$

Table II. Results for the Pseudo-Laplacian (20)

The results show the highly improved accuracy by using the Pseudo-Laplace operator instead of the Neumann boundary value problem. These results once more substantiate the usefulness of our treatment in [18].

5 The unsteady case

For an analysis let denote by v^{n+1}, q^{n+1} the exact velocity and pressure at the time level t_{n+1}. From the pressure correction step (10) we obtain

$$\frac{\beta_0}{\Delta t}\left(u^{n+1} - v^{n+1} + v^{n+1} - \tilde{u}^{n+1}\right) = \nabla\left(q^{n+1} - p^{n+1} + \overline{p}_l^{n+1} - q^{n+1}\right) \text{ in } \Omega.$$

Hence it follows

$$u^{n+1} - v^{n+1} + \frac{\Delta t}{\beta_0}\nabla\left(p^{n+1} - q^{n+1}\right) = \Delta t\left(\frac{\tilde{u}^{n+1} - v^{n+1}}{\Delta t} + \frac{1}{\beta_0}\nabla(\overline{p}_l^{n+1} - q^{n+1})\right).$$

Since the intermediate velocity $\tilde{u}^{n+1}$ is determined by a $k-th$ order time differentation scheme and the extrapolated pressure $\overline{p}_l^{n+1}$ by a $l-th$ order scheme we finally obtain that the error function

$$u^{n+1} - v^{n+1} + \frac{\Delta t}{\beta_0}\nabla(p^{n+1} - q^{n+1}) \tag{25}$$

behaves as

$$\Delta t \left(O(\Delta t^k) + O(\Delta t^l) \right) \; = \; O(\Delta t^{k+1}) + O(\Delta t^{l+1}).$$

Now by taking $k = l = r$ we finally obtain that (25) behaves as

$$O(\Delta t^{r+1}) \;\; \text{for } \Delta t \to 0.$$

Hence by combining a first order backward Euler scheme with a first order extrapolation (i.e., $\overline{p}_1^{n+1} = p^n$) we totally obtain a second order method in time. For $k = l = 2$ we have a third order time discretization. This theoretical result was also confirmed by numerical calculations. In the nonstationary case we consider the following example where

$$\begin{aligned} u_1(x,y,t) &= z(t)r(\overline{x})s(\overline{y}), \\ u_2(x,y,t) &= -z(t)s(\overline{x})r(\overline{y}), \\ p(x,y,t) &= w(t)exp(\overline{x} + \overline{y} - 1), \end{aligned}$$

and

$$\begin{aligned} z(t) &= (2\pi - 1 + \sin(2\pi t))/2\pi, \\ w(t) &= \cos 2\pi t. \end{aligned}$$

A similar example was already investigated in [19] where a $2nd$-order backward Euler scheme (for the viscous term) was combined with an Adams-Bashforth type evaluation of the convective term. This scheme is preferable to other schemes for its better stability properties (see [23]). It results in a second order time discretization (see [19]). But there was no comparable splitting. Here we really consider the splitted scheme. For the above examples we calculated the *mean quadratic errors*:

$$\begin{aligned} E_{U1} &= \max\{ERU1: \; t \geq 0\} \\ E_{U2} &= \max\{ERU2: \; t \geq 0\} \\ E_p &= \max\{ERP: \; t \geq 0\}, \end{aligned}$$

where $ERU1$, $ERU2$, ERP denote the quadratic errors in u_1, u_2, p at the time level t. Here we only consider the discretization by the Pseudo-Laplace operator. The calculations have been performed for $N = 8$ and $\Delta t = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$. Since the starting-up procedure requires modified lower order schemes we measure the above mean quadratic errors when the effect of initialization has disappeared. Then the errors become periodic in time (the functions z, w are periodic). The corresponding numerical results are given in the tables III, IV.

Δt	$E_{U1} = E_{U2}$	*ratio*	E_p	*ratio*
$\frac{1}{16}$	$4.13 \cdot 10^{-3}$		$1.04 \cdot 10^{-1}$	
$\frac{1}{32}$	$1.16 \cdot 10^{-3}$	3.56	$3.79 \cdot 10^{-2}$	2.74
$\frac{1}{64}$	$3.03 \cdot 10^{-4}$	3.83	$1.19 \cdot 10^{-2}$	3.18
$\frac{1}{128}$	$7.62 \cdot 10^{-5}$	3.98	$3.12 \cdot 10^{-3}$	3.81
$\frac{1}{256}$	$1.91 \cdot 10^{-5}$	3.99	$7.85 \cdot 10^{-4}$	3.97

Table III. E_{U1}, E_{U2} and E_p for $r = 1$

Δt	$E_{U1} = E_{U2}$	*ratio*	E_p	*ratio*
$\frac{1}{16}$	$1.78 \cdot 10^{-3}$		$5.05 \cdot 10^{-2}$	
$\frac{1}{32}$	$1.63 \cdot 10^{-4}$	10.9	$7.34 \cdot 10^{-3}$	6.88
$\frac{1}{64}$	$2.00 \cdot 10^{-5}$	8.15	$8.32 \cdot 10^{-4}$	8.82
$\frac{1}{128}$	$2.50 \cdot 10^{-6}$	8.00	$1.03 \cdot 10^{-4}$	8.08
$\frac{1}{256}$	$3.13 \cdot 10^{-7}$	7.99	$1.29 \cdot 10^{-5}$	7.98

Table IV. E_{U1}, E_{U2} and E_p for $r = 2$

The ratio of the errors show that we really obtain a second order method for $r = 1$ and a third order method for $r = 2$. The same orders were also obtained in the discrete H^1-norm and the maximum norm. Therefore the numerical results confirm the theoretical results for $r = 1$ and $r = 2$. We also tried the above splitting for $r \geq 3$ but we observed instability. The time splitting is diverging for $r \geq 3$. Only a stable third order method is available. So the main result of our paper is the third order convergence of our splitting where an extrapolation technique for the pressure has been used. Unfortunately, the scheme becomes unstable for the fourth order time discretization. Anyway it is good to know that we have obtained a third order method. Then by taking the step size Δt sufficiently small we obtain a high accuracy in each time step.

By further numerical computations we found that the instability for $r \geq 3$ is due to the pressure correction step (10 - 12) where an extrapolant for the pressure has been used. A direct Uzawa decoupling procedure does not lead to a similar instability phenomen. We now consider the spectral Stokes system (18 - 19). Let us introduce the following abbreviations:

$$\begin{aligned} L_0 &= -D^0_\Delta + \frac{\beta_0}{\Delta t} I, \\ A &= D^0_x L_0^{-1} D_x + D^0_y L_0^{-1} D_y. \end{aligned}$$

Then a direct Uzawa decoupling leads to the following equation for the pressure p^{n+1}:

$$A p^{n+1} = h^{n+1}, \tag{26}$$

where

$$h^{n+1} = D^0_x L_0^{-1} \hat{f}_1^{n+1} + D^0_y L_0^{-1} \hat{f}_2^{n+1}.$$

After solving (26) for p^{n+1} the velocity components u_1^{n+1}, u_2^{n+1} can easily be determined by solving the two Helmholtz equations:

$$L_0 u_1^{n+1} = \hat{f}_1^{n+1} - D_x p^{n+1} \tag{27}$$

$$L_0 u_2^{n+1} = \hat{f}_2^{n+1} - D_y p^{n+1}. \tag{28}$$

Now one has to solve two Helmholtz equations for L_0 and one equation for A. In contrast to the previous splitting the Pseudo-Laplace operator B is now changed to A. Clearly, in the limit $\Delta t \to 0$ we obtain:

$$\Delta t \to 0: \quad A \to \frac{\Delta t}{\beta_0} B.$$

But for fixed Δt the inversion of A requires one additional solution of an equation for L_0. Hence for fixed Δt one should use an already calculated $LU-$decomposition of L_0. For decreasing Δt the iterative procedure and preconditioner for the matrix $\frac{\Delta t}{\beta_0} B$ should also yield a good solver for the operator A. Since there is no pressure correction step the order of the discretization is reduced by one, i.e., the error (25) behaves as

$$O(\Delta t^r) \text{ for } \Delta t \to 0.$$

This was fully confirmed by numerical experiments for our nonstationary example. We calculated the errors E_{U1}, E_{U2}, and E_p for $N = 12$ and $\Delta t = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$. The corresponding numerical results for $r = 6$ are given in table V. We observe that there is no instability for $r \leq 6$. For $r \geq 7$ the method is diverging. So the early instability for the above pressure correction scheme is caused by the pressure correction step and the extrapolation for the pressure. Therefore if one is interested in a very high order time discretization one should use the above sixth order Uzawa splitting scheme.

Δt	$E_{U1} = E_{U2}$	*ratio*	E_p	*ratio*
$\frac{1}{16}$	$1.38 \cdot 10^{-7}$		$2.77 \cdot 10^{-7}$	
$\frac{1}{32}$	$2.23 \cdot 10^{-9}$	61.88	$4.48 \cdot 10^{-9}$	61.83
$\frac{1}{64}$	$3.52 \cdot 10^{-11}$	63.35	$7.06 \cdot 10^{-11}$	63.46
$\frac{1}{128}$	$5.50 \cdot 10^{-13}$	64.00	$1.11 \cdot 10^{-12}$	63.60

Table V. E_{U1}, E_{U2} and E_p for $r = 6$

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