A canonical rate-independent model of geometrically linear isotropic gradient plasticity with isotropic hardening and plastic spin accounting for the Burgers vector

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Abstract

In this paper we propose a canonical variational framework for rate-independent phenomenological geometrically linear gradient plasticity with plastic spin. The model combines the additive decomposition of the total distortion into non-symmetric elastic and plastic distortions, with a defect energy contribution taking account of the Burgers vector through a dependence only on the dislocation density tensor Curl p giving rise to a non-symmetric nonlocal backstress, and isotropic hardening response only depending on the accumulated equivalent plastic strain. The model is fully isotropic and satisfies linearized gauge-invariance conditions, i.e., only true state-variables appear. The model satisfies also the principle of maximum dissipation which allows to show existence for the weak formulation. For this result, a recently introduced Korn's inequality for incompatible tensor fields is necessary. Uniqueness is shown in the class of strong solutions. For vanishing energetic length scale, the model reduces to classical elasto-plasticity with symmetric plastic strain ε_p and standard isotropic hardening.

Key words: plasticity, gradient plasticity, variational modeling, dissipation function, geometrically necessary dislocations, incompatible distortions, rate-independent models, isotropic hardening, generalized standard material, variational inequality, convex analysis, associated flow rule, defect energy, dislocation density, plastic rotation, global dissipation inequality, Burgers vector, plastic spin.

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1. Introduction

Since the celebrated work of Tresca [113], classical plasticity has been cast within the years into a beautiful framework in which both theoretical and computational aspects were examined (see e.g. [66, 71, 7, 106, 55, 110, 22]). Even perfect classical plasticity has been recently revived by [22, 34, 35] with the use of the energetic approach for rate-independent processes developed by [74, 75].

On the other hand, a number of experimental results have shown size-dependencies for the material behaviour in small scales (micron/meso) (see e.g. [30, 108]). However, classical plasticity models are scale independent and therefore cannot capture those size-effects. This has

led in the last thirty years to an abundant literature ([1, 2, 78, 31, 5, 41, 39, 43, 45, 32, 33, 101]) on theories of gradient plasticity with the aim of accommodating the experimentally observed size effects mentioned above. The so-called *energetic and dissipative length scales* have been involved. Moreover, effort has also been made in the past years to provide mathematical results for the initial boundary values problems and inequalities describing some models of gradient plasticity (see for instance, [24, 103, 26, 83, 27, 38, 91, 92, 28]). Several contributions on the computational aspects have been made as well ([25, 90, 14, 104]).

In most of the above-mentioned models of gradient plasticity, the plastic rotation has been ignored. If a polycrystal is treated as a randomly oriented collection of grains, it is clear that the plastic distortion p, which must then be seen as the average slip over all glide planes, will be non-symmetric. Therefore, plastic spin is a reality also in polycrystalline modelling. The situation is less clear when one aims at an overall effective phenomenological description in which individual glide planes are not resolved. It is possible to show that in a purely local isotropic theory the plastic spin can be suppressed without loss of generality. The situation is again different in gradient-plasticity extensions, in which it is generally agreed that plastic spin is automatically included (e.g. [41]). However, no agreement has been reached on how to precisely include the effect of plastic spin. Our contribution aims at proposing a canonical framework to do exactly this. In [41, 11, 12, 100] models discussing the role of the plastic rotation have been proposed. For instance, [100] discusses the need to incorporate the plastic rotation in an isotropic gradient plasticity framework in order to capture some effects of a crystallographic model for a large collection of grains in a polycrystal. In the mathematical context, existence results for models with plastic spin have also been obtained ([83, 27, 28]).

The modelling challenge which we faced in the past can be explained as follows. Given the additive decomposition of the total non-symmetric distortion (the displacement gradient ∇u), is it possible to write down a model with plastic spin (the plastic distortion p is not symmetric) and allow for a defect energy depending on Nye's dislocation density tensor Curl p together with an isotropic hardening response which is, however, only driven by the accumulated equivalent plastic strain $\gamma_p = \int_0^t \|\text{sym} \, \dot{p}\| \, ds = \int_0^t \|\dot{\varepsilon}_p\| \, ds$, and cast all that in the suitable convex variational framework of the principle of maximum dissipation? In Section 3 we present exactly such a model. Our previous attempts of modelling in this direction were based on the (rate-explicit) dual flow rule but failed to satisfy the principle of maximum dissipation, [46, p. 454], see also [48, 97, 19, 49].[1]

The new model proposed in this paper, which involves only one energetic length scale L_c has some features which make it stand out from other proposals in rate-independent gradient plasticity with plastic spin as:

- it allows for plastic spin in a most transparent manner: for vanishing characteristic energetic length scale $L_c \to 0$, the plastic spin vanishes as well and the model turns into classical elasto-plasticity with symmetric plastic strain $\varepsilon_p = \text{sym} \, p$ and with isotropic hardening based only on the accumulated equivalent plastic strain $\gamma_p = \int_0^t \|\text{sym} \, \dot{p}\| \, ds = \int_0^t \|\dot{\varepsilon}_p\| \, ds$;

- it is completely isotropic and (linearized) frame-indifferent;

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[1] It is often assumed that the plastic evolution $\dot{p}$ associated with a state of yield maximizes the dissipation relative to all admissible states. This is also equivalent to I'liushin's postulate ([67]).

- it is (linearized) *gauge-invariant*: this means that it satisfies invariance under compatible transformations of the reference system, i.e., in the linearized context it is invariant under

$$
\boxed{
\begin{array}{rcl}
\nabla u(x) & \longrightarrow & \nabla u(x) + \nabla \vartheta_p(x) \quad \forall \vartheta_p \in C^1(\mathbb{R}^3, \mathbb{R}^3) \\
p(x) & \longrightarrow & p(x) + \nabla \vartheta_p(x)
\end{array}
}
,
$$

  which is also known as translational T(3)-gauge invariance ([63, 64, 65, 29]);

- it contains only properly defined state-variables ([105, 29]). In this context, notice that, as mentioned in De Wit [23, p.1478]: "... the plastic strain [sym p] is not a state quantity, i.e., it cannot be determined from the [current] state of the body." Through a proper definition of infinitesimal state-variables, this will be clearly presented in [29].

In this model, the hardening type response is depending on a (nonlocal) kinematic term which is the non-symmetric backstress contribution $\mu\, L_c^2\, \mathrm{Curl\, Curl}\, p$, solely responsible for the appearance of plastic spin or not and related to the geometrically necessary dislocation (GND) density distribution. The isotropic hardening is related to statistically stored dislocations (SSD), which take into account a "plastically homogeneous" effect as they accumulate already during a macroscopically homogeneous deformation. Here, the SSD evolution is modelled by two isotropic hardening variables $\gamma_p = \int_0^t \|\mathrm{sym}\, \dot{p}\|\, ds$ and $\omega_p = \int_0^t \|\mathrm{skew}\, \dot{p}\|\, ds$. Hence, the full plastic distortion, and not only its symmetric part, may contribute to hardening. This is in accordance with the physical nature of plastic flow since also the evolution of the skew-symmetric part of p indicates dislocation motion. It is noteworthy that classical linear Prager-type kinematical hardening cannot be accommodated in the "state-variable" approach adopted here since the corresponding backstress contribution $\varepsilon_p = \mathrm{sym}\, p$ as such is not a state-variable (see e.g. [95, 105]).

Notwithstanding the use of the dislocation density tensor $\mathrm{Curl}\, p$, we claim that our model is properly isotropic. In passing, notice that taking $\mathrm{Curl}\, \varepsilon_p = \mathrm{Curl\, sym}\, p$ is physically inadmissible since $\mathrm{Curl}\, \varepsilon_p$ is not a defect measure for $\varepsilon_p \in \mathrm{Sym}(3)$. Rather, one should then take Kröner's incompatibility tensor $\mathrm{inc}\, \varepsilon_p := \mathrm{Curl}[(\mathrm{Curl}\, \varepsilon_p)^T]$. The possibilities to do exactly this will be explained in the forthcoming paper [29]. On the other hand, claims in the recent literature [107][2] that dependencies of a model on the dislocation density tensor $\mathrm{Curl}\, p$ exclude isotropy will also be critically examined in [29].

It is sometimes argued that plastic spin is irrelevant in the case of isotropy ([61]).[3] The question whether one needs a theory with plastic spin is just the question whether one can work with a symmetric plastic strain tensor ε_p as the only variable in a phenomenological plasticity theory. Our development clearly shows that claims such as in [61] are unfounded and seem to indicate that there are different notions involved of what isotropy precisely means. This subject will also be discussed further in [29].

We call *spin cross-hardening* the situation where plastic flow in the plastic strain $\varepsilon_p = \mathrm{sym}\, p$ causes hardening in the plastic rotation evolution of skew p and vice-versa. In our model we see

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[2]Steigmann and Gupta [107, p.410] put forward that: "... the dislocation density [tensor] $\mathrm{Curl}\, p$ is well-defined under symmetry transformations only if the symmetry group is discrete." From that they conclude that it is not possible to obtain an isotropic plasticity model including $\mathrm{Curl}\, p$.

[3]Krishnan and Steigmann [61, p.722] argue that plastic spin associated with a flow rule for plastic evolution can be suppressed in the isotropic case without loss of generality. We understand that this is only true for the local theory, i.e., $L_c = 0$ as confirmed in [46, p.511].

that spin cross-hardening does not take place, i.e., only the accumulated equivalent plastic strain influences hardening in the evolution of the plastic strain and only the accumulated equivalent plastic rotation influences hardening in the evolution of the plastic rotation.

A remark concerning the mathematical treatment of single crystal plasticity is also in order. First, it is clear that such a theory is also a phenomenological model, albeit on a different scale. In the single crystal case the assumption of different glide systems lead to an immediate anisotropy of plastic flow and plastic spin is automatically included. However, the dislocation density contribution, when looked at it in detail, leads to a full gradient control of the plastic slip on each glide-plane. Therefore, the nonsymmetric plastic distortion p, which is the combined plastic slip on each glide plane, is automatically controlled in the standard Sobolev space $H^1(\Omega)$ ([102, 14]). By contrast, our isotropic framework means to give up detailed control of the plastic distortion due to additional invariance conditions that have to be respected. The effect is that there is not even an immediate $H(\mathrm{Curl})$-control of the plastic distortion. Therefore, the mathematically more challenging model is, without any doubt, the isotropic dislocation-based model with plastic spin treated here.

Notice that there are some similarities between our new isotropic model and the early one proposed by Gurtin [41]. In fact, both models share: a complete isotropic formulation, decoupled evolution equations into symmetric and skew-symmetric rates (isotropic hardening possibly coming from both), a dissipation depending also on plastic spin, the same defect and elastic energies, only an energetic length scale connected to the dislocation density tensor and both reduce to classical plasticity when the energetic length scale is zero. Now, there are also nontrivial differences between the two models. In fact, the model in [41] is visco-plastic, includes local nonsymmetric kinematical backstress due to dissipative viscoplastic hardening, it is not cast into a variational framework and does not have existence results so far. Also the model in [41] involves a novel microforce balance as well as boundary conditions on the moving elastic-plastic boundary[4] and a dissipation function depending also on the gradient of the plastic distortion rate (see also [93]). The type of dissipation function considered in our model leads to an elastic region with Tresca-like branches and hence, in the flow rule in rate-explicit dual form, we get a case distinction to determine on which part of the yield surface the evolution takes place. In this, there are therefore similarities to crystal plasticity in which each glide plane has its own evolution and stresses are projected to the glide planes (see e.g. [40]). In our model the non-symmetric Eshelby-type stress Σ_E driving the plastic evolution is projected on $\mathfrak{sl}(3) \cap \mathrm{Sym}(3)$ (symmetric and traceless tensors) for the plastic strain evolution and $\mathfrak{so}(3)$ (skew-symmetric tensors) for the plastic spin evolution.

Let us emphasize that, while we will present the complete and rigorous mathematical existence theory to our model, the main thrust in this work is not only of analytical nature. It rather consists also in presenting that modeling framework for plastic spin which we deem to be the most suited one.

This paper is now structured as follows. In Section 2, we present some notations and def-

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[4]Unlike [41], no novel microforce balance needs to be introduced in our model. Also, in our theory, nonstandard (tangential) boundary conditions for the plastic distortion p are always defined at the external boundary of the material only and the question on how to define them at a moving elastic-plastic boundary never arises. Any specific prescription of such boundary conditions at the elastic-plastic boundary could be in contradiction with the uniqueness result which we obtain for strong solutions.

initions. In Section 3, we introduce various aspects of the model, in particular, the flow rule in both primal and dual formulations with the key role played by the dissipation function. In Sections 4 and 5, we study mathematical aspects (existence and uniqueness) of the model while in Section 6, we recover the classical plasticity framework when the characteristic length scale is set to be zero ($L_c \to 0$).

2. Some notational agreements and definitions

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega$, which is occupied by the elastoplastic body in its undeformed configuration. Let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. A material point in Ω is denoted by x and the time domain under consideration is the interval $[0, T]$.

For every $a, b \in \mathbb{R}^3$, we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $|a|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3}$. We denote by $\mathbb{R}^{3\times3}$ the set of real 3×3 tensors. The standard Euclidean scalar product on $\mathbb{R}^{3\times3}$ is given by $\langle A, B \rangle_{\mathbb{R}^{3\times3}} = \text{tr}\left[AB^T\right]$, where B^T denotes the transpose tensor of B. Thus, the Frobenius tensor norm is $\|A\|^2 = \langle A, A \rangle_{\mathbb{R}^{3\times3}}$. In the following we omit the subscripts $\mathbb{R}^3$ and $\mathbb{R}^{3\times3}$. The identity tensor on $\mathbb{R}^{3\times3}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(A) = \langle A, \mathbb{1} \rangle$. The set $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3\times3} \,|\, X^T = -X\}$ is the Lie-Algebra of skew-symmetric tensors. We let $\text{Sym}\,(3) := \{X \in \mathbb{R}^{3\times3} \,|\, X^T = X\}$ denote the vector space of symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3\times3} \,|\, \text{tr}\,(X) = 0\}$ be the Lie-Algebra of traceless tensors. For every $X \in \mathbb{R}^{3\times3}$, we set $\text{sym}(X) = \frac{1}{2}\left(X + X^T\right)$, $\text{skew}\,(X) = \frac{1}{2}\left(X - X^T\right)$ and $\text{dev}(X) = X - \frac{1}{3}\text{tr}\,(X)\,\mathbb{1} \in \mathfrak{sl}(3)$ for the symmetric part, the skew-symmetric part and the deviatoric part of X, respectively. Quantities which are constant in space will be denoted with an overbar, e.g., $\overline{A} \in \mathfrak{so}(3)$ for the function $A : \mathbb{R}^3 \to \mathfrak{so}(3)$ which is constant with constant value $\overline{A}$.

The body is assumed to undergo infinitesimal deformations. Its behaviour is governed by a set of equations and constitutive relations. Below is a list of variables and parameters used throughout the paper:

- u is the displacement of the macroscopic material points;

- p is the infinitesimal plastic distortion variable which is a non-symmetric second order tensor, incapable of sustaining volumetric changes; that is, $p \in \mathfrak{sl}(3)$. The tensor p represents the average plastic slip; p is not a state-variable, while the rate $\dot{p}$ is;

- $e = \nabla u - p$ is the infinitesimal elastic distortion which is a non-symmetric second order tensor and is a state-variable;

- $\varepsilon_p = \text{sym}\,p$ is the symmetric infinitesimal plastic strain tensor, which is also trace free, $\varepsilon_p \in \mathfrak{sl}(3)$; ε_p is not a state-variable; the rate $\dot{\varepsilon}_p = \text{sym}\,\dot{p}$ is a state-variable;

- $\text{skew}\,p$ is called plastic rotation or plastic spin;

- $\varepsilon_e = \text{sym}\,(\nabla u - p)$ is the symmetric infinitesimal elastic strain tensor and is a state-variable;

- σ is the Cauchy stress tensor which is a symmetric second order tensor and is a state-variable;

- σ_0 and $\widehat{\sigma}_0$ are the initial yield stresses for plastic strain and plastic spin, respectively and both are state-variables;

- σ_{y} and $\widehat{\sigma}_{\mathrm{y}}$ are the current yield stresses for plastic strain and plastic spin, respectively and both are state-variables;

- f is the body force;

- $\operatorname{Curl} p = -\operatorname{Curl} e = \alpha$ is the dislocation density tensor satisfying the so-called Bianchi identities $\operatorname{Div} \alpha = 0$ and is a state-variable;

- $\gamma_p = \int_0^t \|\operatorname{sym} \dot{p}\|\, ds = \int_0^t \|\dot{\varepsilon}_p\|\, ds$ is the accumulated equivalent plastic strain and is a state-variable;

- $\omega_p = \int_0^t \|\operatorname{skew} \dot{p}\|\, ds$ is the accumulated equivalent plastic rotation and is a state-variable;

- $\int_0^t \sqrt{\dot{\gamma}_p^2 + \dot{\omega}_p^2}\, ds = \int_0^t \|\dot{p}\|\, ds$ represents the accumulated equivalent plastic distortion which is a state-variable.

For isotropic media, the fourth order isotropic elasticity tensor $\mathbb{C}_{\mathrm{iso}} : \operatorname{Sym}(3) \to \operatorname{Sym}(3)$ is given by

$$\mathbb{C}_{\mathrm{iso}} \operatorname{sym} X = 2\mu \operatorname{dev} \operatorname{sym} X + \kappa \operatorname{tr}(X)\mathbb{1} = 2\mu \operatorname{sym} X + \lambda \operatorname{tr}(X)\mathbb{1} \tag{2.1}$$

for any second-order tensor X, where μ and λ are the Lamé moduli satisfying

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0, \tag{2.2}$$

and $\kappa > 0$ is the bulk modulus. These conditions suffice for pointwise positive definiteness of the elasticity tensor in the sense that there exists a constant $m_0 > 0$ such that

$$\forall X \in \mathbb{R}^{3\times 3} : \quad \langle \operatorname{sym} X, \mathbb{C}_{\mathrm{iso}} \operatorname{sym} X \rangle \geq m_0 \|\operatorname{sym} X\|^2. \tag{2.3}$$

The space of square integrable functions is $L^2(\Omega)$, while the Sobolev spaces used in this paper are:

$$
\begin{aligned}
\mathrm{H}^1(\Omega) &= \{u \in L^2(\Omega) \mid \operatorname{grad} u \in L^2(\Omega)\}, & \operatorname{grad} &= \nabla, \\
&\quad \|u\|_{\mathrm{H}^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\operatorname{grad} u\|_{L^2(\Omega)}^2, & \forall u &\in \mathrm{H}^1(\Omega), \\
\mathrm{H}(\operatorname{curl};\Omega) &= \{v \in L^2(\Omega) \mid \operatorname{curl} v \in L^2(\Omega)\}, & \operatorname{curl} &= \nabla\times, \\
&\quad \|v\|_{\mathrm{H}(\operatorname{curl};\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \|\operatorname{curl} v\|_{L^2(\Omega)}^2, & \forall v &\in \mathrm{H}(\operatorname{curl};\Omega).
\end{aligned}
\tag{2.4}
$$

For every $X \in C^1(\Omega, \mathbb{R}^{3\times 3})$ with rows X^1, X^2, X^3, we use in this paper the definition of $\operatorname{Curl} X$ in [83, 111]:

$$\operatorname{Curl} X = \begin{pmatrix} \operatorname{curl} X^1 & - & - \\ \operatorname{curl} X^2 & - & - \\ \operatorname{curl} X^3 & - & - \end{pmatrix} \in \mathbb{R}^{3\times 3}, \tag{2.5}$$

for which Curl $\nabla v = 0$ for every $v \in C^2(\Omega, \mathbb{R}^3)$. Notice that the definition of Curl X above is such that $(\text{Curl } X)^T a = \text{curl}(X^T a)$ for every $a \in \mathbb{R}^3$ and this clearly corresponds to the transpose of the Curl of a tensor as defined in [43, 46].

The following function spaces and norms will also be used later.

$$
\begin{aligned}
\text{H(Curl; } \Omega, \mathbb{R}^{3\times3}) &= \left\{ X \in L^2(\Omega, \mathbb{R}^{3\times3}) \mid \text{Curl } X \in L^2(\Omega, \mathbb{R}^{3\times3}) \right\}, \\
\|X\|^2_{\text{H(Curl;}\Omega)} &= \|X\|^2_{L^2(\Omega)} + \|\text{Curl } X\|^2_{L^2(\Omega)}, \quad \forall X \in \text{H(Curl; } \Omega, \mathbb{R}^{3\times3}), \qquad (2.6) \\
\text{H(Curl; } \Omega, \mathbb{E}) &= \left\{ X : \Omega \to \mathbb{E} \mid X \in \text{H(Curl; } \Omega, \mathbb{R}^{3\times3}) \right\},
\end{aligned}
$$

for $\mathbb{E} := \mathfrak{sl}(3)$ or $\text{Sym}(3) \cap \mathfrak{sl}(3)$.

We also consider the space

$$
\text{H}_0(\text{Curl; } \Omega, \Gamma, \mathbb{R}^{3\times3}) \tag{2.7}
$$

as the completion in the norm in (2.6) of the space $\left\{ q \in C^\infty(\Omega, \Gamma, \mathbb{R}^{3\times3}) \mid q \times n|_\Gamma = 0 \right\}$. Therefore, this space generalizes the tangential Dirichlet boundary condition

$$
q \times n|_\Gamma = 0
$$

to be satisfied by the plastic distortion p or the plastic strain $\varepsilon^p := \text{sym } p$. The space

$$
\text{H}_0(\text{Curl; } \Omega, \Gamma, \mathbb{E})
$$

is defined as in (2.6).

The divergence operator Div on second order tensor-valued functions is also defined row-wise as

$$
\text{Div } X = \begin{pmatrix} \text{div } X_1 \\ \text{div } X_2 \\ \text{div } X_3 \end{pmatrix}. \tag{2.8}
$$

3. The description of the model

3.1. The balance equation

The conventional macroscopic force balance leads to the equation of equilibrium

$$
\text{div } \sigma + f = \mathbf{0} \tag{3.1}
$$

in which σ is the infinitesimal symmetric Cauchy stress and f is the body force.

3.2. Constitutive equations.

The constitutive equations are obtained from a free energy imbalance together with a flow law that characterizes plastic behaviour. Since the model under study involves plastic spin by which we mean that the plastic distortion p is not symmetric, we consider directly an additive

decomposition of the displacement gradient ∇u into elastic and plastic components e and p, so that

$$\nabla u = e + p \,, \tag{3.2}$$

with the nonsymmetric plastic distortion p incapable of sustaining volumetric changes; that is,

$$\operatorname{tr}(p) = \operatorname{tr}(\operatorname{sym} p) = \operatorname{tr}(\varepsilon_p) = 0 \,. \tag{3.3}$$

Here, $\varepsilon_e = \operatorname{sym} e = \operatorname{sym}(\nabla u - p)$ is the infinitesimal elastic strain and $\varepsilon_p = \operatorname{sym} p$ is the plastic strain while $\operatorname{sym} \nabla u = (\nabla u + \nabla u^T)/2$ is the total strain.

We consider a free energy in the additively separated form

$$\Psi(\nabla u, \operatorname{Curl} p, \gamma_p, \omega_p) : \quad = \quad \underbrace{\Psi_e^{\mathrm{lin}}(\operatorname{sym} e)}_{\text{elastic energy}} \quad + \quad \underbrace{\Psi_{\mathrm{curl}}^{\mathrm{lin}}(\operatorname{Curl} p)}_{\text{defect energy (GND)}} \tag{3.4}$$
$$+ \quad \underbrace{\Psi_{\mathrm{iso}}(\gamma_p, \omega_p)}_{\text{hardening energy (SSD)}} \quad ,$$

where

$$\Psi_e^{\mathrm{lin}}(\operatorname{sym} e) := \tfrac{1}{2} \langle \operatorname{sym} e, \mathbb{C}_{\mathrm{iso}} \operatorname{sym} e \rangle, \qquad \Psi_{\mathrm{curl}}^{\mathrm{lin}}(\operatorname{Curl} p) := \tfrac{1}{2} \mu L_c^2 \|\operatorname{Curl} p\|^2 \,,$$
$$\Psi_{\mathrm{iso}}(e_p, w_p) := \tfrac{1}{2} \mu \alpha_1 |\gamma_p|^2 + \tfrac{1}{2} \mu \alpha_2 |\omega_p|^2 \,. \tag{3.5}$$

Here, $L_c \geq 0$ is an energetic length scale which characterizes the contribution of the defect energy density to the system, α_1 and α_2 are positive nondimensional isotropic hardening constants, γ_p and ω_p are isotropic hardening variables. The defect energy is conceptually related to geometrically necessary dislocations (GND). It is formed by the long-ranging stress-fields of excess dislocations and may be recovered by appropriate inelastic deformation. The isotropic hardening energy is related to statistically stored dislocations (SSD).[5] It is formed by the local stress-fields of all dislocations and can only be recovered in thermodynamical processes such as annealing, recrystallization or chemical reactions.

3.2.1. The derivation of the dissipation inequality

The local free-energy imbalance states that

$$\dot{\Psi} - \langle \sigma, \dot{e} \rangle - \langle \sigma, \dot{p} \rangle \leq 0 \,. \tag{3.6}$$

Now we expand the first term, substitute (3.4) and get

$$\langle \mathbb{C}_{\mathrm{iso}} \operatorname{sym} e - \sigma, \operatorname{sym} \dot{e} \rangle - \langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \operatorname{Curl} p, \operatorname{Curl} \dot{p} \rangle + \mu \alpha_1 \gamma_p \dot{\gamma}_p + \mu \alpha_2 \omega_p \dot{\omega}_p \leq 0 \,, \tag{3.7}$$

---

[5]It is an easy matter to generalize the defect-energy contribution as well as the elasticity relation to the complete anisotropic setting. However, this does not add anything to enhance understanding of the paper and hence we leave these easy generalizations aside.

which, using arguments from thermodynamics gives the elastic relation

$$\sigma = \mathbb{C}_{\mathrm{iso}} \operatorname{sym} e = 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr}(\nabla u - p)\mathbb{1} \tag{3.8}$$

and the local reduced dissipation inequality

$$-\langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \operatorname{Curl} p, \operatorname{Curl} \dot{p} \rangle + \mu \, \alpha_1 \gamma_p \, \dot{\gamma}_p + \mu \, \alpha_2 \, \omega_p \, \dot{\omega}_p \leq 0. \tag{3.9}$$

Now we integrate (3.9) over Ω and get

$$
\begin{aligned}
0 \;\geq\; & \int_\Omega \Big[-\langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \operatorname{Curl} p, \operatorname{Curl} \dot{p} \rangle + \mu \, \alpha_1 \gamma_p \, \dot{\gamma}_p + \mu \, \alpha_2 \, \omega_p \, \dot{\omega}_p \Big] \\
=\; & -\int_\Omega \Big[\langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \operatorname{Curl} \operatorname{Curl} p, \dot{p} \rangle + \mu \, \alpha_1 \gamma_p \, \dot{\gamma}_p + \mu \, \alpha_2 \, \omega_p \, \dot{\omega}_p \\
& + \sum_{i=1}^{3} \operatorname{div}\Big(\mu L_c^2 \, \frac{d}{dt} p^i \times (\operatorname{Curl} p)^i \Big) \Big] \, dx \,.
\end{aligned}
\tag{3.10}
$$

Using the divergence theorem we obtain

$$
\begin{aligned}
\int_\Omega \Big[\langle -\sigma + \mu L_c^2 \operatorname{Curl} \operatorname{Curl} p, \dot{p} \rangle + \mu \, \alpha_1 \gamma_p \, \dot{\gamma}_p + \mu \, \alpha_2 \, \omega_p \, \dot{\omega}_p \Big] \, dx & \\
+ \sum_{i=1}^{3} \int_{\partial\Omega} \mu L_c^2 \langle \dot{p}^i \times (\operatorname{Curl} p)^i, n \rangle da & \leq 0 \,.
\end{aligned}
\tag{3.11}
$$

In order to obtain a dissipation inequality in the spirit of classical plasticity, we assume that the infinitesimal plastic distortion p satisfies the so-called *linearized insulation condition*

$$\sum_{i=1}^{3} \int_{\partial\Omega} \mu L_c^2 \, \langle \frac{d}{dt} p^i \times (\operatorname{Curl} p)^i, n \rangle dS = 0 \,. \tag{3.12}$$

Under (3.12) and splitting the rates orthogonally in the scalar product $\langle \cdot, \cdot \rangle$,

$$\dot{p} = \operatorname{sym} \dot{p} + \operatorname{skew} \dot{p} \,, \tag{3.13}$$

we then obtain a global version of the reduced dissipation inequality[6]

$$
\begin{aligned}
\int_\Omega [\langle \sigma + \Sigma_{\mathrm{curl}}^{\mathrm{lin}}, \dot{p} \rangle + g_1 \, \dot{\gamma}_p + g_2 \, \dot{\omega}_p] \, dx & \geq 0 \,, \\
\Leftrightarrow \quad \int_\Omega [\langle \sigma + \operatorname{sym} \Sigma_{\mathrm{curl}}^{\mathrm{lin}}, \operatorname{sym} \dot{p} \rangle + \langle \operatorname{skew} \Sigma_{\mathrm{curl}}^{\mathrm{lin}}, \operatorname{skew} \dot{p} \rangle + g_1 \, \dot{\gamma}_p + g_2 \, \dot{\omega}_p] \, dx & \geq 0 \,,
\end{aligned}
\tag{3.14}
$$

where

$$\Sigma_{\mathrm{curl}}^{\mathrm{lin}} := -\mu L_c^2 \operatorname{Curl} \operatorname{Curl} p \,, \qquad g_1 := -\mu \, \alpha_1 \, \gamma_p \,, \qquad g_2 := -\mu \, \alpha_2 \, \omega_p \,. \tag{3.15}$$

---

[6]Gurtin [41, p.4] refers to Menzel and Steinmann [73] and writes: "... but [they] satisfy the dissipation inequality [only] globally."

For further use we define the non-symmetric Eshelby-type stress tensor driving the plastic evolution

$$\Sigma_E := \sigma + \Sigma_{\text{curl}}^{\text{lin}}, \tag{3.16}$$

with the non-symmetry relating only to the nonlocal term $\Sigma_{\text{curl}}^{\text{lin}}$. In terms of Σ_E the global reduced dissipation inequality can be expressed as

$$\int_\Omega [\langle \operatorname{dev} \operatorname{sym} \Sigma_E, \operatorname{sym} \dot{p} \rangle + \langle \operatorname{skew} \Sigma_E, \operatorname{skew} \dot{p} \rangle + g_1\, \dot{\gamma}_p + g_2\, \dot{\omega}_p]\, dx \geq 0. \tag{3.17}$$

The split used in (3.13) is a constitutive choice in that it will suggest a suitable format on how to satisfy the inequality (3.14) in all deformation processes. In our previously proposed models (see [27]), this split has not been used.

3.2.2. The boundary conditions on the plastic distortion

The condition (3.12) is satisfied if we assume for instance that the boundary is a perfect conductor. This means that the tangential component of p vanishes on $\partial\Omega$. In the context of dislocation dynamics these conditions express the requirement that there is no flux of the Burgers vector across a hard boundary. Gurtin [41] and also Gurtin and Needleman [42] introduce the following different types of boundary conditions for the plastic distortion

$$\begin{aligned}
\partial_t p \times n|_{\Gamma_{\text{hard}}} &= 0 \quad \text{"micro-hard" (perfect conductor)} \\
\partial_t p|_{\Gamma_{\text{hard}}} &= 0 \quad \text{"hard-slip" \quad (in the context of crytal plasticity)} \\
\operatorname{Curl} \partial_t p \times n|_{\Gamma_{\text{hard}}} &= 0 \quad \text{"micro-free" .}
\end{aligned} \tag{3.18}$$

We specify a sufficient condition for the micro-hard boundary condition, namely

$$p \times n|_{\Gamma_{\text{hard}}} = 0 \tag{3.19}$$

and assume for simplicity only $\Gamma_{\text{hard}} = \partial\Omega = \Gamma$. Note that this boundary condition constrains the plastic slip in tangential direction only, which is what we expect to happen at the physical boundary Γ_{hard}.

3.3. The flow rule

3.3.1. The flow rule in its primal formulation

Let $D : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$D(s, t) := \sqrt{\sigma_0^2\, s^2 + \widehat{\sigma}_0^2\, t^2}, \tag{3.20}$$

where $\sigma_0, \widehat{\sigma}_0 > 0$ are the initial yield stresses for symmetric strain $\operatorname{sym} p$ and skew-symmetric spin $\operatorname{skew} p$, respectively.[7]

---

[7]Both values together will define the elastic domain in the stress space and this domain must have nonempty interior. Therefore, we need $\sigma_0, \widehat{\sigma}_0 > 0$. Without isotropic hardening the elastic domain turns out to be $\{\Sigma_E \in \mathbb{R}^{3\times3} \mid \|\operatorname{dev} \operatorname{sym} \Sigma_E\| \leq \sigma_0,\ \|\operatorname{skew} \Sigma_E\| \leq \widehat{\sigma}_0\}$.

We consider the dissipation function Δ defined by [8]

$$\Delta(q,\eta,\beta) := \begin{cases} D(\|\mathrm{sym}\,q\|, \|\mathrm{skew}\,q\|) & \text{if } \|\mathrm{sym}\,q\| \le \eta \quad \text{and} \quad \|\mathrm{skew}\,q\| \le \beta\,, \\ \infty & \text{otherwise}\,. \end{cases} \tag{3.21}$$

We could consider a more general dissipation function corresponding e.g. to the function

$$\widehat{D}(s,t) := r_1\,s + r_2\,t + \sqrt{\sigma_0^2\,s^2 + \widehat{\sigma}_0^2\,t^2} \qquad \text{with } r_1, r_2 \ge 0\,. \tag{3.22}$$

Such a choice will not add any particular feature to the current model. In fact, as shown in the appendix this simply corresponds to the expansion of the initial elastic domain (i.e. before isotropic hardening takes place).

The flow rule in its primal formulation can be derived using the principle of the minimum of the dissipation function [48, 97, 19], stating that the rate of the internal variables is the minimizer of a functional L consisting of the sum of the rate of the free energy and the dissipation function with respect to appropriate boundary conditions,

$$L = \int_\Omega [\dot{\Psi} + \Delta]dx. \tag{3.23}$$

The principle of the minimum of the dissipation function is closely related to the principle of maximum dissipation. Both are not physical principles but thermodynamically consistent selection rules which turn out to be convenient if no other information is available or if existing flow rules are to be extended to a more general situation. For a detailed investigation, see [49]. A very general exposition for coupled physical processes is worked out in [53, 54]. Applications to the evolution of plastic microstructures can be found in [50, 52, 51, 59].

Employing a partial integration, the stationarity conditions of (3.23) can be compactly stated as

$$\Sigma_p \in \partial\Delta(\dot{\Gamma}_p) \qquad \text{where } \Sigma_p = (\sigma + \Sigma_{\mathrm{curl}}^{\mathrm{lin}}, g_1, g_2) \quad \text{and} \quad \Gamma_p = (p, \gamma_p, \omega_p)\,, \tag{3.24}$$

and where $\partial\Delta$ denotes the subdifferential of Δ. That is, for $\Sigma_p \in \partial\Delta(\dot{\Gamma}_p)$ we must have

$$\begin{aligned} \Delta(\overline{\Gamma}) &\ge \Delta(\dot{\Gamma}_p) + \langle \Sigma_p, \overline{\Gamma} - \dot{\Gamma}_p \rangle \\ &\ge \Delta(\dot{\Gamma}_p) + \langle \sigma + \Sigma_{\mathrm{curl}}^{\mathrm{lin}}, q - \dot{p} \rangle + g_1\,(\eta - \dot{\gamma}_p) + g_2\,(\beta - \dot{\omega}_p) \\ &= \Delta(\dot{\Gamma}_p) + \langle \Sigma_E, q - \dot{p} \rangle + g_1\,(\eta - \dot{\gamma}_p) + g_2\,(\beta - \dot{\omega}_p)\,, \end{aligned} \tag{3.25}$$

for every $\overline{\Gamma} = (q,\eta,\beta)$. By choosing $\overline{\Gamma} = (0,0,0)$ in (3.25), we get the reduced dissipation inequality in pointwise form

$$\begin{aligned} &\langle \Sigma_E, \dot{p} \rangle + g_1\,\dot{\gamma}_p + g_2\,\dot{\omega}_p \ge 0\,, \\ &\Leftrightarrow \quad \langle \mathrm{dev}\,\mathrm{sym}\,\Sigma_E, \mathrm{sym}\,\dot{p} \rangle + \langle \mathrm{skew}\,\Sigma_E, \mathrm{skew}\,\dot{p} \rangle + g_1\,\dot{\gamma}_p + g_2\,\dot{\omega}_p] \ge 0 \end{aligned} \tag{3.26}$$

---

[8]Gurtin [41, p.2554] notes: "One would expect that, plastically, the material response to spin differs to straining, and that straining and spin each incur dissipation." Gurtin's choice of the dissipation function in [41] corresponds to $\widehat{\sigma}_0 = \chi\,\sigma_0 \ge 0$ in (3.20). Also, Gurtin [41, p.2558] takes $\chi \to 0$ formally and recovers classical plasticity. If we want to take $\widehat{\sigma}_0 \to 0$ in our setting, then we encounter a problem described in Section 5.4.

3.3.2. The flow law in its dual formulation

While the flow rule in the primal formulation is extremely condensed and will allow us a mathematical treatment (existence), we need the representation of the flow rule in the dual formulation in most computational implementations and for the uniqueness proof in Section 4.4. For this formulation of the flow rule we need to derive the set of admissible (generalized) stresses $\mathcal{E}$ (the elastic domain) corresponding to the dissipation function Δ. According to the principle of maximum dissipation,[9] the flow law in dual form is formulated in the context of convex analysis as

$$\dot{\Gamma}_p \in N_{\mathcal{E}}(\Sigma_p) \quad \Leftrightarrow \quad \langle \dot{\Gamma}_p, \overline{\Sigma} - \Sigma_p \rangle \leq 0 \qquad \forall \overline{\Sigma} \in \mathcal{E}, \tag{3.27}$$

where $N_{\mathcal{E}}(\Sigma_p)$ is the normal cone to the set $\mathcal{E}$ of admissible stresses at Σ_p. Therefore, we need to find the set $\mathcal{E}$. In the context of convex analysis, the indicator function $I_{\mathcal{E}}$ of the set $\mathcal{E}$ is the Fenchel-Legendre conjugate of the dissipation function Δ. Let us find the set $\mathcal{E}$ whose interior $\text{int}(\mathcal{E})$ is the elastic domain and its boundary $\partial \mathcal{E}$ is the yield surface.

For $\Sigma_p = (\Sigma_E, g_1, g_2)$ with $\Sigma_E := \sigma + \Sigma_{\text{curl}}^{\text{lin}}$, we have

$$
\begin{aligned}
I_{\mathcal{E}}(\Sigma_p) &= \sup\{ \langle \Sigma_p, \Gamma \rangle - \Delta(\Gamma) \mid \Gamma = (q, \eta, \beta) \} \\
&= \sup\{ \langle \Sigma_E, q \rangle + g_1\,\eta + g_2\,\beta - \Delta(q, \eta, \beta) \mid \|\text{sym}\,q\| \leq \eta, \;\; \|\text{skew}\,q\| \leq \beta \} \\
&= \sup_q \left[\sup_{\eta, \beta}\{ \langle \Sigma_E, q \rangle + g_1\,\eta + g_2\,\beta - \Delta(q, \eta, \beta) \mid \|\text{sym}\,q\| \leq \eta, \;\; \|\text{skew}\,q\| \leq \beta \} \right] \\
&= \sup_q \{ \langle \Sigma_E, q \rangle + g_1\,\|\text{sym}\,q\| + g_2\,\|\text{skew}\,q\| - \Delta(q, \|\text{sym}\,q\|, \|\text{skew}\,q\|) \} \\
&= \sup_q \left\{ \begin{array}{l} \langle \text{dev}\,\text{sym}\,\Sigma_E, \text{sym}\,q \rangle + \langle \text{skew}\,\Sigma_E, \text{skew}\,q \rangle \\ \quad + g_1\,\|\text{sym}\,q\| + g_2\,\|\text{skew}\,q\| - \Delta(q, \|\text{sym}\,q\|, \|\text{skew}\,q\|) \end{array} \right\},
\end{aligned}
\tag{3.28}
$$

where the supremum with respect to η and β is achieved for $\eta = \|\text{sym}\,q\|$ and $\beta = \|\text{skew}\,q\|$ since $g_1 \leq 0$ and $g_2 \leq 0$.

Now taking the supremum with respect to q and using the fact that $\langle \Sigma_E, q \rangle$ is maximum with respect to q only when q is in the direction of Σ_E, we find that it is not restrictive to assume that

$$\text{sym}\,q = s\,\frac{\text{dev}\,\text{sym}\,\Sigma_E}{\|\text{dev}\,\text{sym}\,\Sigma_E\|} \quad \text{and} \quad \text{skew}\,q = t\,\frac{\text{skew}\,\Sigma_E}{\|\text{skew}\,\Sigma_E\|}. \tag{3.29}$$

We then obtain

$$I_{\mathcal{E}}(\Sigma_p) = \sup_{s \geq 0,\, t \geq 0} \left\{ s\,(\|\text{dev}\,\text{sym}\,\Sigma_E\| + g_1) + t\,(\|\text{skew}\,\Sigma_E\| + g_2) - \sqrt{\sigma_0^2\,s^2 + \widehat{\sigma}_0^2\,t^2} \right\}. \tag{3.30}$$

To simplify the function of s and t to be maximized in (3.30), we set

$$A := \|\text{dev}\,\text{sym}\,\Sigma_E\| + g_1 \quad \text{and} \quad B := \|\text{skew}\,\Sigma_E\| + g_2, \tag{3.31}$$

and hence,

$$I_{\mathcal{E}}(\Sigma_p) = \sup_{s \geq 0,\, t \geq 0} \left\{ A\,s + B\,t - \sqrt{\sigma_0^2\,s^2 + \widehat{\sigma}_0^2\,t^2} \right\}. \tag{3.32}$$

---

[9]which, again, is not a principle, but a useful and often made simplifying assumption.

Notice immediately that

$$I_{\mathcal{E}}(\Sigma_p) = \sup_{s \geq 0,\, t \geq 0} \left\{ \frac{A}{\sigma_0}s + \frac{B}{\widehat{\sigma}_0}t - \sqrt{s^2 + t^2} \right\}. \tag{3.33}$$

The following holds (see Appendix A for details)

$$\sup_{s \geq 0,\, t \geq 0} \left\{ \frac{A}{\sigma_0}s + \frac{B}{\widehat{\sigma}_0}t - \sqrt{s^2 + t^2} \right\} = \begin{cases} 0 & \text{if } \begin{cases} A \leq \sigma_0 & \text{if } B \leq 0 \\[4pt] B \leq \widehat{\sigma}_0 & \text{if } A \leq 0 \\[4pt] \dfrac{A^2}{\sigma_0^2} + \dfrac{B^2}{\widehat{\sigma}_0^2} \leq 1 & \text{if } \begin{cases} A \geq 0 \\ B \geq 0 \end{cases} \end{cases} \\[30pt] \infty & \text{otherwise}. \end{cases} \tag{3.34}$$

Let us now introduce a set $\mathcal{K} \subset \mathbb{R}^2$ needed for elucidating the branching behaviour of our flow rule and defined by

$$\mathcal{K} := \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3, \tag{3.35}$$

where

$$\begin{aligned} \mathcal{K}_1 &= \left(-\infty,\, \sigma_0 \right] \times \left(-\infty,\, 0 \right], & \mathcal{K}_2 &= \left(-\infty,\, 0 \right] \times \left(-\infty, \widehat{\sigma}_0 \right], \\ \mathcal{K}_3 &= \left\{ (A, B) \in \mathbb{R}^+ \times \mathbb{R}^+ \;\middle|\; \frac{A^2}{\sigma_0^2} + \frac{B^2}{\widehat{\sigma}_0^2} \leq 1 \right\}. \end{aligned}$$

The set $\mathcal{K}$ in the AB-plane is represented graphically in Figure 1. Notice that the set $\mathcal{K}$ itself is **not** the elastic domain. In our setting, the elastic domain is then defined as the interior of the set

$$\mathcal{E} = \left\{ (\Sigma_E, g_1, g_2) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^- \times \mathbb{R}^- \;\middle|\; (\|\mathrm{dev\,sym}\,\Sigma_E\| + g_1, \|\mathrm{skew}\,\Sigma_E\| + g_2) \in \mathcal{K} \right\}. \tag{3.36}$$

In other terms, the set $\mathcal{E}$, which is also called the set of admissible stresses, is expressed as

$$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3, \tag{3.37}$$

where

$$\begin{aligned} \mathcal{E}_1 &= \left\{ (\Sigma_E, g_1, g_2) \;\middle|\; \|\mathrm{dev\,sym}\,\Sigma_E\| \leq -g_1 + \sigma_0, \quad \|\mathrm{skew}\,\Sigma_E\| \leq -g_2 \right\}, \\[4pt] \mathcal{E}_2 &= \left\{ (\Sigma_E, g_1, g_2) \;\middle|\; \|\mathrm{dev\,sym}\,\Sigma_E\| \leq -g_1, \quad \|\mathrm{skew}\,\Sigma_E\| \leq -g_2 + \widehat{\sigma}_0 \right\}, \\[4pt] \mathcal{E}_3 &= \begin{cases} (\Sigma_E, g_1, g_2) \;\middle|\; \|\mathrm{dev\,sym}\,\Sigma_E\| \geq -g_1, \quad \|\mathrm{skew}\,\Sigma_E\| \geq -g_2, \\[8pt] \qquad \text{and} \quad \dfrac{(\|\mathrm{dev\,sym}\,\Sigma_E\| + g_1)^2}{\sigma_0^2} + \dfrac{(\|\mathrm{skew}\,\Sigma_E\| + g_2)^2}{\widehat{\sigma}_0^2} \leq 1 \end{cases}. \end{aligned}$$
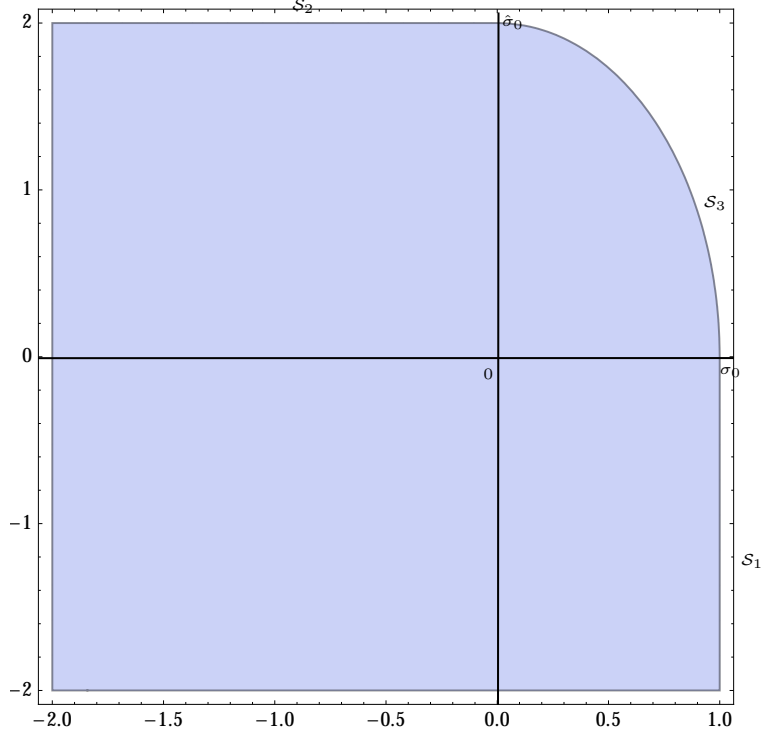
Figure 1: The set $\mathcal{K}$ in the AB-plane. On $\mathcal{S}_1$ the flow is only driven by the symmetric rate part $\operatorname{sym}\dot{p}$ as $\operatorname{skew}\dot{p} = 0$. On $\mathcal{S}_1$ the flow is only driven by the skew-symmetric rate part $\operatorname{skew}\dot{p}$ as $\operatorname{sym}\dot{p} = 0$. On $\mathcal{S}_3$ the flow is driven by both symmetric and skew-symmetric rate parts $\operatorname{skew}\dot{p}$ and $\operatorname{sym}\dot{p}$.

Hence, the yield surface is given by

$$\partial\mathcal{E} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \tag{3.38}$$

with

$$
\begin{aligned}
\mathcal{S}_1 &= \left\{ (\Sigma_E, g_1, g_2) \mid \|\operatorname{dev}\operatorname{sym}\Sigma_E\| = -g_1 + \sigma_0\,, \quad \|\operatorname{skew}\Sigma_E\| \leq -g_2 \right\}, \\
\mathcal{S}_2 &= \left\{ (\Sigma_E, g_1, g_2) \mid \|\operatorname{dev}\operatorname{sym}\Sigma_E\| \leq -g_1\,, \quad \|\operatorname{skew}\Sigma_E\| = -g_2 + \widehat{\sigma}_0 \right\}, \quad (3.39) \\
\mathcal{S}_3 &= \left\{ \begin{aligned} &(\Sigma_E, g_1, g_2) \mid \|\operatorname{dev}\operatorname{sym}\Sigma_E\| \geq -g_1, \quad \|\operatorname{skew}\Sigma_E\| \geq -g_2, \\ &\quad \text{and} \quad \frac{(\|\operatorname{dev}\operatorname{sym}\Sigma_E\| + g_1)^2}{\sigma_0^2} + \frac{(\|\operatorname{skew}\Sigma_E\| + g_2)^2}{\widehat{\sigma}_0^2} = 1 \end{aligned} \right\}.
\end{aligned}
$$

Let us briefly discuss the evolution of the yield surface. The hardening behavior will depend on the values of the moduli α_1, α_2, and on the location of the generalized stress state (Σ_E, g_1, g_2) on the yield surface. There are four different possibilities, displayed in Fig. 2. For clarity, we summarize the various cases in Table 1.

	$\alpha_1 = 0,\ \alpha_2 = 0$	$\alpha_1 > 0,\ \alpha_2 = 0$	$\alpha_1 = 0,\ \alpha_2 > 0$	$\alpha_1 > 0,\ \alpha_2 > 0$
$(\Sigma_E, g_1, g_2) \in \mathcal{S}_1$	a.)	b.)	a.)	b.)
$(\Sigma_E, g_1, g_2) \in \mathcal{S}_2$	a.)	a.)	c.)	c.)
$(\Sigma_E, g_1, g_2) \in \mathcal{S}_3$	a.)	b.)	c.)	d.)

Table 1: Evolution of the yield surface, different hardening scenarios.

Next, our goal is to present a strong and a weak formulation of the model, followed by two existence results for which there is an important distinction between the cases $\alpha_2 > 0$ and $\alpha_2 = 0$ in the free-energy Ψ in (3.4)-(3.5).

4. The complete mathematical formulation in the case $\alpha_2 > 0$

In this section, we present the full description of the model in the case $\alpha_2 > 0$ in the free-energy Ψ in (3.4)-(3.5) as well as a corresponding existence result. The case $\alpha_2 > 0$ means that there is always isotropic hardening in the spin-evolution equation. We recall that the dissipation function Δ is given in (3.21) and the yield function in the case $\sigma_0 > 0$ and $\widehat{\sigma}_0 > 0$ (see (3.39)) is given by

$$
\phi(\Sigma_p) := \begin{cases}
\|\operatorname{dev} \operatorname{sym} \Sigma_E\| + g_1 - \sigma_0 & \text{on } \mathcal{S}_1 \\[2mm]
\|\operatorname{skew} \Sigma_E\| + g_2 - \widehat{\sigma}_0 & \text{on } \mathcal{S}_2 \\[2mm]
\dfrac{(\|\operatorname{dev} \operatorname{sym} \Sigma_E\| + g_1)^2}{\sigma_0^2} + \dfrac{(\|\operatorname{skew} \Sigma_E\| + g_2)^2}{\widehat{\sigma}_0^2} - 1 & \text{on } \mathcal{S}_3 \,.
\end{cases}
\tag{4.1}
$$

4.1. The strong formulation

The strong formulation of the model consists in finding:

(i) the displacement $u \in \mathsf{H}^1(0, T; \mathsf{H}_0^1(\Omega, \Gamma, \mathbb{R}^3))$,

(ii) the infinitesimal plastic distortion $p \in \mathsf{H}^1(0, T; L^2(\Omega, \mathfrak{sl}(3)))$ with

$$
\operatorname{Curl} p \in \mathsf{H}^1(0, T; L^2(\Omega, \mathbb{R}^{3\times3})) \quad \text{and} \quad \operatorname{Curl} \operatorname{Curl} p \in \mathsf{H}^1(0, T; L^2(\Omega, \mathbb{R}^{3\times3}))\,,
$$

(iii) the internal isotropic hardening variables $\gamma_p,\ \omega_p \in \mathsf{H}^1(0, T; L^2(\Omega))$

such that the content of Table 2 holds.

4.2. The weak formulation

Assume that the problem in Section 4.1 has a solution $(u, p, \gamma_p, \omega_p)$. Let $v \in H^1(\Omega, \mathbb{R}^3)$ with $v_{|\Gamma_D} = 0$. Multiply the equilibrium equation with $v - \dot{u}$ and integrate in space by parts and use
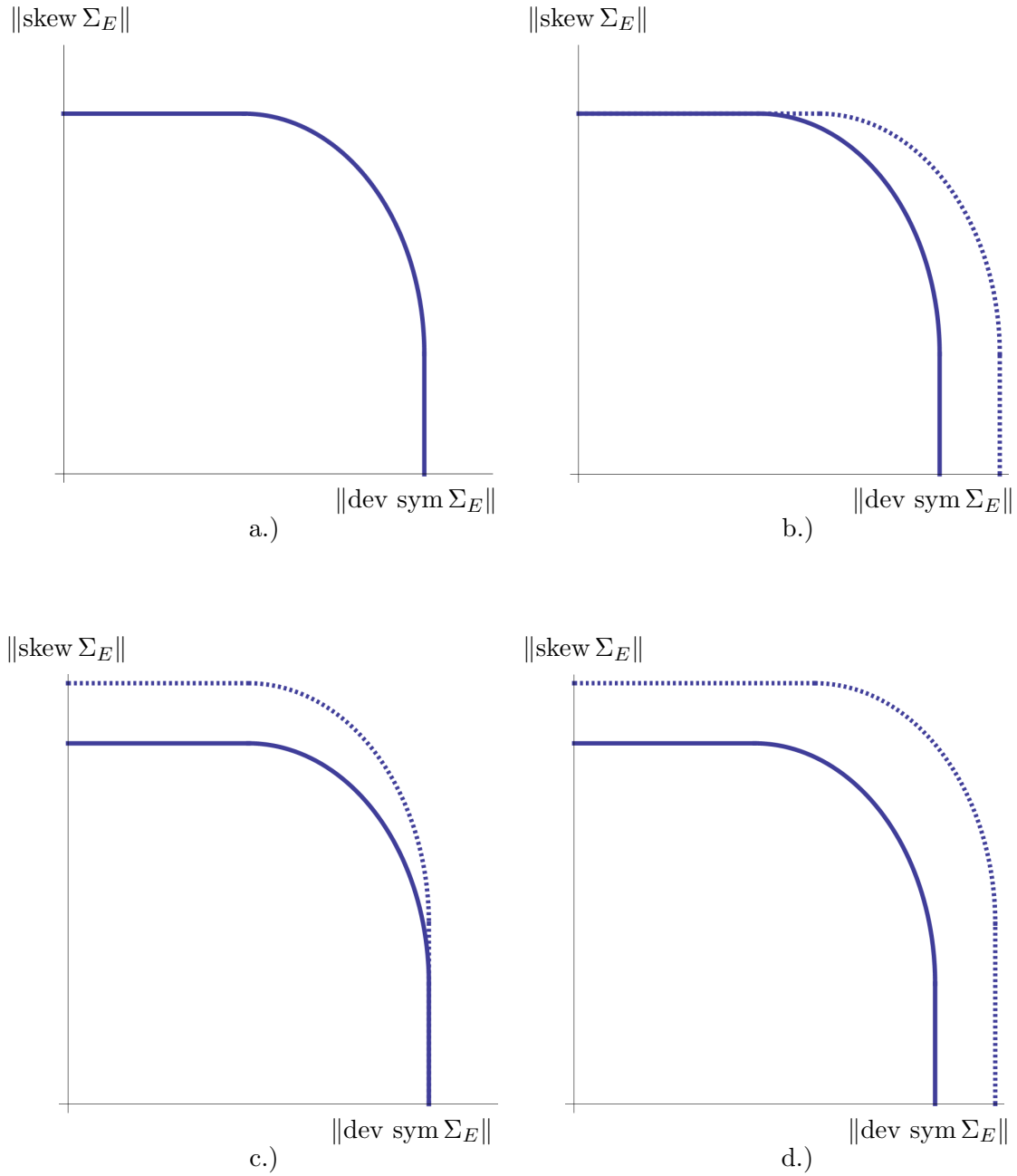
Figure 2: Evolution of the yield surface by isotropic hardening: original yield surface depicted by solid line, evolved yield surface by dashed line. Different cases: a.) no evolution of yield surface, b.) expansion in direction of $\|\text{dev sym}\,\Sigma_E\|$, c.) expansion in direction of $\|\text{skew}\,\Sigma_E\|$, d.) expansion in both directions.

Additive split of distortion:	$\nabla u = e + p, \quad \varepsilon^e = \mathrm{sym}\,e, \quad \varepsilon^p = \mathrm{sym}\,p$
Equilibrium:	$\mathrm{Div}\,\sigma + f = 0$ with $\sigma = \mathbb{C}_{\mathrm{iso}}\varepsilon^e$
Free energy:	$\frac{1}{2}\langle\mathbb{C}_{\mathrm{iso}}\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\mu\,L_c^2\,\|\mathrm{Curl}\,p\|^2 + \frac{1}{2}\mu\,\alpha_1\,|\gamma_p|^2 + \frac{1}{2}\mu\,\alpha_2\,|w_p|^2$
Yield condition:	$\phi(\Sigma_p) = 0$ with ϕ given in (4.1)
where	$\Sigma_p = (\Sigma_E, g_1, g_2), \quad \Sigma_E := \sigma + \Sigma^{\mathrm{lin}}_{\mathrm{curl}}, \quad \Sigma^{\mathrm{lin}}_{\mathrm{curl}} = -\mu\,L_c^2\,\mathrm{Curl}\,\mathrm{Curl}\,p$ $g_1 = -\mu\,\alpha_1\gamma_p, \quad g_2 = -\mu\,\alpha_2\,w_p$
Dissipation inequality:	$\int_\Omega [\langle\mathrm{dev}\,\mathrm{sym}\,\Sigma_E, \mathrm{sym}\,\dot{p}\rangle + \langle\mathrm{skew}\,\Sigma_E, \mathrm{skew}\,\dot{p}\rangle + g_1\,\dot{\gamma}_p + g_2\,\dot{w}_p]\,dx \geq 0$
Dissipation function:	$\Delta(\dot{\Gamma}_p)$ is defined in (3.21)
Flow law in primal form:	$\Sigma_p \in \partial\Delta(\dot{\Gamma}_p)$
Flow law in dual form on $\mathcal{S}_1$:	$\begin{cases} \mathrm{sym}\,\dot{p} = \lambda\dfrac{\mathrm{dev}\,\mathrm{sym}\,\Sigma_E}{\|\mathrm{dev}\,\mathrm{sym}\,\Sigma_E\|} \quad \dot{\gamma}_p = \lambda = \|\mathrm{sym}\,\dot{p}\| \\ \mathrm{skew}\,\dot{p} = 0 \qquad\qquad\qquad \dot{\omega}_p = 0 \end{cases}$
Flow law in dual form on $\mathcal{S}_2$:	$\begin{cases} \mathrm{sym}\,\dot{p} = 0 \qquad\qquad\qquad \dot{\gamma}_p = 0 \\ \mathrm{skew}\,\dot{p} = \lambda\dfrac{\mathrm{skew}\,\Sigma_E}{\|\mathrm{skew}\,\Sigma_E\|} \quad \dot{\omega}_p = \lambda = \|\mathrm{skew}\,\dot{p}\| \end{cases}$
Flow law in dual form on $\mathcal{S}_3$:	$\begin{cases} \mathrm{sym}\,\dot{p} = \dfrac{2\lambda}{\sigma_0^2}\,(\|\mathrm{dev}\,\mathrm{sym}\,\Sigma_E\| + g_1)\dfrac{\mathrm{dev}\,\mathrm{sym}\,\Sigma_E}{\|\mathrm{dev}\,\mathrm{sym}\,\Sigma_E\|} \\[2ex] \mathrm{skew}\,\dot{p} = \dfrac{2\lambda}{\widehat{\sigma}_0^2}\,(\|\mathrm{skew}\,\Sigma_E\| + g_2)\dfrac{\mathrm{skew}\,\Sigma_E}{\|\mathrm{skew}\,\Sigma_E\|} \\[2ex] \dot{\gamma}_p = \dfrac{2\lambda}{\sigma_0^2}\,(\|\mathrm{dev}\,\mathrm{sym}\,\Sigma_E\| + g_1) = \|\mathrm{sym}\,\dot{p}\| \\[2ex] \dot{\omega}_p = \dfrac{2\lambda}{\widehat{\sigma}_0^2}\,(\|\mathrm{skew}\,\Sigma_E\| + g_2) = \|\mathrm{skew}\,\dot{p}\| \\[2ex] \|\dot{p}\| = \sqrt{\dot{\gamma}_p^2 + \dot{\omega}_p^2}, \qquad \lambda = \dfrac{1}{2}\sqrt{\sigma_0^2\,\dot{\gamma}_p^2 + \widehat{\sigma}_0^2\,\dot{\omega}_p^2} \end{cases}$
KKT conditions:	$\lambda \geq 0, \quad \phi(\Sigma_E, g_1, g_2) \leq 0, \quad \lambda\,\phi(\Sigma_E, g_1, g_2) = 0$
Boundary conditions for p:	$p \times n = 0$ on $\Gamma, \quad (\mathrm{Curl}\,p) \times n = 0$ on $\partial\Omega \setminus \Gamma$
Function space for p:	$p(t, \cdot) \in \mathrm{H}(\mathrm{Curl};\,\Omega,\,\mathbb{R}^{3\times 3})$

Table 2: The model with isotropic hardening and plastic spin: the case $\alpha_2 > 0$. Because of the control of the L^2-norm of both isotropic hardening variables η and β and the constraints $\|\mathrm{sym}\,q\| \leq \eta$ and $\|\mathrm{skew}\,q\| \leq \beta$, the function space for the non-symmetric plastic distortion p is easily seen to be $\mathrm{H}(\mathrm{Curl};\,\Omega,\,\mathbb{R}^{3\times 3})$. Note that the dual formulation of the flow rule needs a case distinction depending on the branches $\mathcal{S}_i$ of the yield surface while the primal formulation does not need it. Clearly, isotropic hardening for the plastic strain depends only on the accumulated equivalent plastic strain and isotropic hardening for the plastic rotation depends only on the accumulated equivalent plastic rotation. Therefore, there is no spin cross-hardening. In this flow rule in dual form we appreciate a Tresca like behaviour (see [56]) in that we have to use a case distinction to determine on which part of the yield surface the evolution takes place.

the symmetry of σ and the elasticity relation to get

$$\int_\Omega \langle\mathbb{C}_{\mathrm{iso}}\,\mathrm{sym}(\nabla u - p), \mathrm{sym}(\nabla v - \nabla\dot{u})\rangle\,dx = \int_\Omega f(v - \dot{u})\,dx\,. \tag{4.2}$$

Now, for any $q \in C^\infty(\overline{\Omega}, \mathfrak{sl}(3))$ such that $q \times n = 0$ on Γ_D and any η, $\beta \in L^2(\Omega)$, we integrate (3.25) over Ω, integrate by parts the term with $\mathrm{Curl\,Curl}$ using the boundary conditions

$$(q - \dot{p}) \times n = 0 \text{ on } \Gamma_D, \qquad \mathrm{Curl}\, p \times n = 0 \text{ on } \partial\Omega \setminus \Gamma_D$$

and get

$$\int_\Omega \Delta(\overline{\Gamma})\, dx - \int_\Omega \Delta(\dot{\Gamma}_p)\, dx - \int_\Omega \langle \mathbb{C}_{\mathrm{iso}}(\mathrm{sym}\,\nabla u - \mathrm{sym}\, p), \mathrm{sym}\, q - \mathrm{sym}\, \dot{p} \rangle\, dx$$

$$+ \mu\, L_c^2 \int_\Omega \Big[\langle \mathrm{Curl}\, p, \mathrm{Curl}\,(q - \dot{p}) \rangle + \mu\, \alpha_1\, \gamma_p\, (\eta - \dot{\gamma}_p) + \mu\, \alpha_2\, \omega_p\, (\beta - \dot{\omega}_p) \Big]\, dx \geq 0. \quad (4.3)$$

Now adding up (4.2) and (4.3) we get the following weak formulation of the problem in Section 4.1 in the form of a variational inequality:

$$\int_\Omega \langle \mathbb{C}_{\mathrm{iso}}(\mathrm{sym}\,\nabla u - \mathrm{sym}\, p), (\mathrm{sym}\,\nabla v - \mathrm{sym}\, q) - (\mathrm{sym}\,\nabla \dot{u} - \mathrm{sym}\, \dot{p}) \rangle\, dx \qquad (4.4)$$

$$+ \mu\, L_c^2 \int_\Omega \langle \mathrm{Curl}\, p, \mathrm{Curl}\,(q - \dot{p}) \rangle + \int_\Omega \Big[\mu\, \alpha_1 \gamma_p(\eta - \dot{\gamma}_p) + \mu\, \alpha_2\, \omega_p(\beta - \dot{\omega}_p) \Big]\, dx$$

$$+ \int_\Omega \Delta(\overline{\Gamma})\, dx - \int_\Omega \Delta(\dot{\Gamma}_p)\, dx \geq \int_\Omega f(v - \dot{u})\, dx.$$

4.3. Existence result for the weak formulation

To prove the existence result for the weak formulation (4.4), we closely follow the abstract machinery developed by Han and Reddy in [55] for mathematical problems in geometrically linear classical plasticity and used for instance in [24, 103, 83, 27, 28] for models of gradient plasticity. To this aim, (4.4) is written as the variational inequality of the second kind: find $w = (u, p, \gamma_p, \omega_p) \in \mathsf{H}^1(0, T; Z)$ such that $w(0) = 0$, $\dot{w}(t) \in W$ for a.e. $t \in [0, T]$ and

$$\boldsymbol{a}(\dot{w}, z - w) + j_0(z) - j_0(\dot{w}) \geq \langle \ell, z - \dot{w} \rangle \text{ for every } z \in W \text{ and for a.e. } t \in [0, T], \qquad (4.5)$$

where Z is a suitable Hilbert space and W is some closed, convex subset of Z to be constructed later,

$$\boldsymbol{a}(w, z) = \int_\Omega \Big[\langle \mathbb{C}_{\mathrm{iso}}(\mathrm{sym}\,\nabla u - \mathrm{sym}\, p), \mathrm{sym}\,\nabla v - \mathrm{sym}\, q) \rangle + \mu\, L_c^2 \langle \mathrm{Curl}\, p, \mathrm{Curl}\, q \rangle \quad (4.6)$$

$$+ \mu\, \alpha_1 \gamma_p\, \eta + \mu\, \alpha_2\, \omega_p\, \beta \Big]\, dx,$$

$$j(z) = \int_\Omega \Delta(q, \eta, \beta)\, dx, \qquad (4.7)$$

$$\langle \ell, z \rangle = \int_\Omega f\, v\, dx, \qquad (4.8)$$

for $w = (u, p, \gamma_p, \omega_p)$ and $z = (v, q, \eta, \beta)$ in Z.

The Hilbert space Z and the closed convex subset W are constructed in such a way that the

functionals $\boldsymbol{a}$, j_0 and ℓ satisfy the assumptions in the abstract result in [55, Theorem 7.3]. The key issue here is the coercivity of the bilinear form $\boldsymbol{a}$ on the set W.

We let

$$
\begin{aligned}
\mathsf{V} &= \mathsf{H}_0^1(\Omega, \Gamma_{\mathrm{D}}, \mathbb{R}^3) = \{v \in \mathsf{H}^1(\Omega, \mathbb{R}^3) \mid v_{|\Gamma_{\mathrm{D}}} = 0\}, \\
\mathsf{Q} &= \mathsf{H}_0(\mathrm{Curl}; \Omega, \Gamma, \mathfrak{sl}(3)), \\
\Lambda &= L^2(\Omega), \\
\mathsf{Z} &= \mathsf{V} \times \mathsf{Q} \times \Lambda^2, \\
\mathsf{W} &= \{z = (v, q, \eta, \beta) \in \mathsf{Z} \mid \|\mathrm{sym}\, q\| \leq \eta \text{ and } \|\mathrm{skew}\, q\| \leq \beta\},
\end{aligned}
$$

(4.9)

(4.10)

and define the norms

$$
\begin{aligned}
&\|v\|_V := \|\nabla v\|_{L^2}, \qquad \|q\|_Q := \|q\|_{\mathrm{H}(\mathrm{Curl};\Omega)}, \\
&\|z\|_Z^2 := \|v\|_V^2 + \|q\|_Q^2 + \|\eta\|_{L^2}^2 + \|\beta\|_{L^2}^2 \quad \text{for } z = (v, q, \eta, \beta) \in \mathsf{Z}.
\end{aligned}
$$

(4.11)

Let us show that the bilinear form $\boldsymbol{a}$ is coercive on W. Let therefore $z = (v, q, \eta, \beta) \in \mathsf{W}$.

$$
\begin{aligned}
\boldsymbol{a}(z,z) &\geq m_0 \|\mathrm{sym}\, \nabla v - \mathrm{sym}\, q\|_2^2 \text{ (from (2.3))} + \mu\, L_c^2 \|\mathrm{Curl}\, q\|_2^2 + \mu\, \alpha_1 \|\eta\|_2^2 + \mu\, \alpha_2 \|\beta\|_2^2 \\
&= m_0 \left[\|\mathrm{sym}\, \nabla v\|_2^2 + \|\mathrm{sym}\, q\|_2^2 - 2\langle \mathrm{sym}\, \nabla v, \mathrm{sym}\, p \rangle \right] \\
&\qquad + \mu L_c^2 \|\mathrm{Curl}\, q\|_2^2 + \mu\, \alpha_1 \|\eta\|_2^2 + \mu\, \alpha_2 \|\beta\|_2^2 \\
&\geq m_0 \left[\|\mathrm{sym}\, \nabla v\|_2^2 + \|\mathrm{sym}\, q\|_2^2 - \theta \|\mathrm{sym}\, \nabla v\|_2^2 - \frac{1}{\theta} \|\mathrm{sym}\, q\|_2^2 \right] \quad \text{(Young's inequality)} \\
&\qquad + \mu\, L_c^2 \|\mathrm{Curl}\, q\|_2^2 + \frac{1}{2} \mu\, \alpha_1 \|\eta\|_2^2 + \frac{1}{2} \mu\, \alpha_2 \|\beta\|_2^2 \\
&\qquad + \frac{1}{2} \mu\, \alpha_1 \|\mathrm{sym}\, q\|_2^2 + \frac{1}{2} \mu\, \alpha_1 \|\mathrm{skew}\, q\|_2^2 \quad \text{(using } \|\mathrm{sym}\, q\| \leq \eta, \ \|\mathrm{skew}\, q\| \leq \beta) \\
&= m_0(1-\theta) \|\mathrm{sym}\, \nabla v\|_2^2 + \left[m_0 \left(1 - \frac{1}{\theta}\right) + \frac{1}{2} \mu\, \alpha_1 \right] \|\mathrm{sym}\, q\|_2^2 + \frac{1}{2} \mu\, \alpha_2 \|\mathrm{skew}\, q\|_2^2 \\
&\qquad + \mu\, L_c^2 \|\mathrm{Curl}\, q\|_2^2 + \frac{1}{2} \mu\, \alpha_1 \|\eta\|_2^2 + \frac{1}{2} \mu\, \alpha_2 \|\beta\|_2^2.
\end{aligned}
$$

(4.12)

So, choosing θ such that $\dfrac{m_0}{m_0 + \frac{1}{2} \mu\, \alpha_1} < \theta < 1$, and using Korn's first inequality (see e.g. [79]), there exists some positive constant $C(m_0, \mu, \alpha_1, \alpha_2, L_c, \Omega) > 0$ such that

$$
a(z,z) \geq C \left[\|v\|_V^2 + \|q\|_{\mathrm{H}(\mathrm{Curl};\Omega)}^2 + \|\eta\|_2^2 + \|\beta\|_2^2 \right] = C \|z\|_Z^2 \quad \forall z = (v, q, \eta, \beta) \in \mathsf{W}.
$$

This shows existence for the model with $\alpha_2 > 0$ (and $\alpha_1 > 0$).

4.4. Uniqueness of the strong solution

If in the geometrically linear classical plasticity model with isotropic hardening, the uniqueness of the weak solution is obtained from the formulation in a variational inequality (see [55, Theorem 7.3]) the uniqueness of the weak solution in the context of gradient plasticity with

isotropic hardening has not yet been completely established. However, in some particular cases, the uniqueness has been obtained provided weak solutions are regular enough (see e.g. [55, pp.210-212]). For Prager-type linear kinematical hardening, the uniqueness of strong solutions in infinitesimal perfect gradient plasticity was established in [81]. In our context, we will prove that requiring Curl Curl $p \in L^2(\Omega, \mathbb{R}^{3\times3})$ is enough to guarantee the uniqueness of the strong solution.

In fact, we first notice that if $\Gamma_p = (u, p, \gamma_p, \omega_p) \in \mathsf{W}$ is a solution of (4.4) with Curl Curl $p \in L^2(\Omega, \mathbb{R}^{3\times3})$, then choosing appropriately test functions and integrating by parts, we easily get that $\Gamma_p = (u, p, \gamma_p, \omega_p)$ satisfies the equilibrium equation (3.1) on the one hand and the flow rule in dual form

$$\langle \dot{\Gamma}_p, \overline{\Sigma} - \Sigma_p \rangle \le 0 \qquad \forall \overline{\Sigma} \tag{4.13}$$

on the other hand.

Let us now consider two solutions $\Gamma_{p_i} = (u_i, p_i, \gamma_{p_i}, \omega_{p_i})$, $i = 1, 2$ of (4.4) satisfying the same initial conditions and let $\Sigma_{p_i} = (\Sigma_{E_i}, g_{1_i}, g_{2_i})$ be the corresponding stresses. That is,

$$\Sigma_{E_i} = \sigma_i - \mu\, L_c^2 \operatorname{Curl} \operatorname{Curl} p_i\,, \qquad g_{1_i} = -\mu\, \alpha_1 \gamma_{p_i}\,, \qquad g_{2_i} = -\mu\, \alpha_2\, \omega_{p_i}\,. \tag{4.14}$$

Hence, Γ_{p_i} and Σ_{p_i} satisfy (4.13), that is,

$$\langle \dot{\Gamma}_{p_1}, \overline{\Sigma} - \Sigma_{p_1} \rangle \le 0 \quad \text{and} \quad \langle \dot{\Gamma}_{p_2}, \overline{\Sigma} - \Sigma_{p_2} \rangle \le 0 \qquad \forall \overline{\Sigma} \tag{4.15}$$

Now choose $\overline{\Sigma} = \Sigma_{p_2}$ in $(4.15)_1$ and $\overline{\Sigma} = \Sigma_{p_1}$ in $(4.15)_2$ and add up to get

$$\langle \Sigma_{p_2} - \Sigma_{p_1}, \dot{\Gamma}_{p_1} - \dot{\Gamma}_{p_2} \rangle \le 0\,. \tag{4.16}$$

That is

$$\begin{aligned} \langle \sigma_2 - \sigma_1, \dot{p}_1 - \dot{p}_2 \rangle + \mu\, L_c^2 \langle \operatorname{Curl} \operatorname{Curl}(p_2 - p_1), \dot{p}_2 - \dot{p}_1 \rangle \\ + \mu\, \alpha_1\, (\gamma_{p_2} - \gamma_{p_1})(\dot{\gamma}_{p_2} - \dot{\gamma}_{p_1}) + \mu\, \alpha_2\, (\omega_{p_2} - \omega_{p_1})(\dot{\omega}_{p_2} - \dot{\omega}_{p_1}) \le 0\,. \end{aligned} \tag{4.17}$$

Since σ is symmetric, the latter is equivalent to

$$\begin{aligned} \langle \sigma_2 - \sigma_1, \operatorname{sym}(\dot{p}_1 - \dot{p}_2) \rangle + \mu\, L_c^2 \langle \operatorname{Curl} \operatorname{Curl}(p_2 - p_1), \dot{p}_2 - \dot{p}_1 \rangle \\ + \mu\, \alpha_1\, (\gamma_{p_2} - \gamma_{p_1})(\dot{\gamma}_{p_2} - \dot{\gamma}_{p_1}) + \mu\, \alpha_2\, (\omega_{p_2} - \omega_{p_1})(\dot{\omega}_{p_2} - \dot{\omega}_{p_1}) \le 0\,. \end{aligned} \tag{4.18}$$

Now, substitute $\operatorname{sym} p_i = \operatorname{sym}(\nabla u_i) - \mathbb{C}^{-1}\sigma_i$ obtained from the elasticity relation, into equation (4.18) and get

$$\begin{aligned} \langle \sigma_2 - \sigma_1, \mathbb{C}^{-1}(\dot{\sigma}_2 - \dot{\sigma}_1) \rangle + \mu\, L_c^2 \langle \operatorname{Curl} \operatorname{Curl}(p_2 - p_1), \dot{p}_2 - \dot{p}_1 \rangle + \mu\, \alpha_1\, (\gamma_{p_2} - \gamma_{p_1})(\dot{\gamma}_{p_2} - \dot{\gamma}_{p_1}) \\ + \mu\, \alpha_2\, (\omega_{p_2} - \omega_{p_1})(\dot{\omega}_{p_2} - \dot{\omega}_{p_1}) \le \langle \sigma_1 - \sigma_2, \operatorname{sym}(\nabla \dot{u}_1) - \operatorname{sym}(\nabla \dot{u}_2) \rangle\,. \end{aligned} \tag{4.19}$$

Now, notice that from the equilibrium equation we get

$$\int_\Omega \langle \sigma_1 - \sigma_2, \operatorname{sym}(\nabla \dot{u}_1) - \operatorname{sym}(\nabla \dot{u}_2) \rangle\, dx = 0\,.$$

Hence, for a.e. $t \in (0, T)$, integrate (4.19) over $\Omega \times (0, t)$ then after integrating the term with Curl Curl by parts, we get

$$\int_0^t \frac{d}{ds} \Big[\|\mathbb{C}^{-1/2}(\sigma_2(s) - \sigma_1(s))\|_2^2 + \mu L_c^2 \|\mathrm{Curl}\,(p_1(s) - p_2(s))\|_2^2$$

$$+ \mu \alpha_1 \|\gamma_{p\,2}(s) - \gamma_{p\,1}(s)\|_2^2 + \mu \alpha_2 \|\omega_{p\,1}(s) - \omega_{p\,2}(s)\|_2^2 \Big] ds \leq 0. \quad (4.20)$$

Therefore, we obtain

$$\|(\mathbb{C}^{-1})^{1/2}(\sigma_2 - \sigma_1)\|_2^2 + \mu L_c^2 \|\mathrm{Curl}\,(p_1 - p_2)\|_2^2$$

$$+ \mu \alpha_1 \|\gamma_{p\,2} - \gamma_{p\,1}\|_2^2 + \mu \alpha_2 \|\omega_{p\,2} - \omega_{p\,1}\|_2^2 = 0, \quad (4.21)$$

from which we get $\sigma_1 = \sigma_2$, $\mathrm{Curl}\, p_1 = \mathrm{Curl}\, p_2$, $\gamma_{p\,1} = \gamma_{p\,2}$, $\omega_{p\,2} = \omega_{p\,1}$ and hence, $\Sigma_{E_1} = \Sigma_{E_2}$.

Now, let us prove that $p_1 = p_2$. In fact, from the definition of the normal cone it follows that $\dot{p}_i = 0$ and $\dot{\gamma}_{p\,i} = \dot{\omega}_{p\,i} = 0$ inside the elastic domain $\mathcal{E}$, which from the initial conditions imply that $p_i = 0$ inside $\mathcal{E}$. Now, looking at the flow rule in dual form in Table 2, we easily obtain that $\mathrm{sym}\,\dot{p}_1 = \mathrm{sym}\,\dot{p}_2$ and $\mathrm{skew}\,\dot{p}_1 = \mathrm{skew}\,\dot{p}_2$ on each surface S_k. Therefore, $\dot{p}_1 = \dot{p}_2$ which implies that $p_1 = p_2$ from the initial conditions.

In order to show that $u_1 = u_2$, we use $\mathrm{sym}(\nabla u_i) = \mathbb{C}^{-1}\sigma_i + \mathrm{sym}\,p_i$ obtained from the elasticity relation and get

$$\mathrm{sym}(\nabla(u_1 - u_2)) = \mathbb{C}^{-1}(\sigma_1 - \sigma_2) + \mathrm{sym}(p_1 - p_2) = 0,$$

and hence, from the first Korn's inequality (see e.g. [79]), we get $\nabla(u_1 - u_2) = 0$ which implies that $u_1 = u_2$. Therefore, we finally obtain

$$u_1 = u_2, \qquad \sigma_1 = \sigma_2, \qquad p_1 = p_2, \qquad \gamma_{p\,1} = \gamma_{p\,2}, \qquad \omega_{p\,1} = \omega_{p\,2},$$

and thus the uniqueness of a strong solution to the mathematical problem describing our model of rate-independent geometrically linear gradient plasticity with isotropic hardening and plastic spin in the case $\alpha_2 > 0$, where there is always isotropic hardening in the spin-evolution equation.

4.5. Perfect gradient plasticity with spin

Inspection of the uniqueness proof for strong solutions in Section 4.4 shows that in the case with zero isotropic plastic strain and spin hardening, and the homogeneous boundary conditions $u|_\Gamma = 0$ and $p \times n|_\Gamma = 0$, elastic stresses σ, elastic strains $\varepsilon_e = \mathrm{sym}\,e$ and furthermore elastic distortions $e = \nabla u - p$ are unique. The uniqueness with respect to elastic distortions uses again the new Korn's inequality for incompatible tensor fields [89] since $e \times n|_\Gamma = 0$. In this case, the extra inclusion of the spin and the dislocation density tensor allow to improve uniqueness from elastic strains to elastic distortions.

5. The complete mathematical formulation in the no-spin-hardening case

Here we set $\alpha_2 = 0$ in the free-energy Ψ in (3.4)-(3.5). The case $\alpha_2 = 0$ means that there is no isotropic hardening in the spin-evolution. At present we believe that it is this case which deserves

special attention, since in this model we extend classical plasticity in the weakest possible way to depend on plastic spin. Notably, we do not incur additional spin-hardening. The dissipation function Δ is still the same given in (3.21) and the yield function is given in (4.1). Also, in this model the influence of the SSD's and GND's on plastic flow is neatly separated: the SSD-distribution influences only isotropic hardening through the classical mechanism and the GND-distribution determines the nonlocal kinematic hardening.

5.1. The strong formulation

The strong formulation of the model in this case consists in finding:

(i) the displacement $u \in \mathsf{H}^1(0,T; \mathsf{H}_0^1(\Omega, \Gamma, \mathbb{R}^3))$,

(ii) the infinitesimal plastic distortion p such that $\operatorname{sym} p \in \mathsf{H}^1(0,T; L^2(\Omega, \mathfrak{sl}(3)))$ with

$$\operatorname{Curl} p \in \mathsf{H}^1(0,T; L^2(\Omega, \mathbb{R}^{3\times3})) \quad \text{and} \quad \operatorname{Curl} \operatorname{Curl} p \in \mathsf{H}^1(0,T; L^2(\Omega, \mathbb{R}^{3\times3})),$$

(iii) the internal isotropic hardening variable $\gamma_p \in \mathsf{H}^1(0,T; L^2(\Omega))$

such that the content of Table 3 holds.

5.2. The weak formulation of the model

The weak formulation of the equilibrium equation (3.1) still holds in the case $\alpha_2 = 0$, that is,

$$\int_\Omega \langle \mathbb{C}_{\text{iso}} \operatorname{sym}(\nabla u - p), \operatorname{sym}(\nabla v - \nabla \dot{u})\rangle \, dx = \int_\Omega f(v - \dot{u}) \, dx \qquad \forall v \in H_0^1(\Omega, \mathbb{R}^3). \tag{5.1}$$

Now, the no-spin-hardening version of the primal formulation of the flow rule in (4.3) reads as

$$\int_\Omega \Delta(\overline{\Gamma}) \, dx - \int_\Omega \Delta(\dot{\Gamma}_p) \, dx - \int_\Omega \langle \mathbb{C}_{\text{iso}}(\operatorname{sym} \nabla u - \operatorname{sym} p), \operatorname{sym} q - \operatorname{sym} \dot{p}\rangle \, dx$$

$$+ \mu \, L_c^2 \int_\Omega \Big[\langle \operatorname{Curl} p, \operatorname{Curl}(q - \dot{p})\rangle + \mu \, \alpha_1 \, \gamma_p \, (\eta - \dot{\gamma}_p) \Big] dx \geq 0. \tag{5.2}$$

Adding now (5.1) to (5.2), we obtain the weak formulation of the model in the form of the variational inequality

$$\int_\Omega \langle \mathbb{C}_{\text{iso}}(\operatorname{sym} \nabla u - \operatorname{sym} p), (\operatorname{sym} \nabla v - \operatorname{sym} q) - (\operatorname{sym} \nabla \dot{u} - \operatorname{sym} \dot{p})\rangle \, dx \tag{5.3}$$

$$+ \mu \, L_c^2 \int_\Omega \langle \operatorname{Curl} p, \operatorname{Curl}(q - \dot{p})\rangle + \alpha_1 \, \mu \int_\Omega \gamma_p (\eta - \dot{\gamma}_p) \, dx$$

$$+ \int_\Omega \Delta(\overline{\Gamma}) \, dx - \int_\Omega \Delta(\dot{\Gamma}_p) \, dx \geq \int_\Omega f(v - \dot{u}) \, dx \,.$$

We are now in a position to discuss the existence result in this case.

Additive split of distortion:	$\nabla u = e + p, \quad \varepsilon^e = \operatorname{sym} e, \quad \varepsilon^p = \operatorname{sym} p$
Equilibrium:	$\operatorname{Div} \sigma + f = 0$ with $\sigma = \mathbb{C}_{\mathrm{iso}} \varepsilon^e$
Free energy:	$\frac{1}{2} \langle \mathbb{C}_{\mathrm{iso}} \varepsilon^e, \varepsilon^e \rangle + \frac{1}{2} \mu L_c^2 \|\operatorname{Curl} p\|^2 + \frac{1}{2} \mu \alpha_1 |\gamma_p|^2$
Yield condition:	$\phi(\Sigma_p) = 0$ with ϕ given in (4.1)
where	$\Sigma_p = (\Sigma_E, g_1, g_2), \qquad \Sigma_E := \sigma + \Sigma_{\mathrm{curl}}^{\mathrm{lin}}, \qquad \Sigma_{\mathrm{curl}}^{\mathrm{lin}} = -\mu L_c^2 \operatorname{Curl} \operatorname{Curl} p$ $g_1 = -\mu \alpha_1 \gamma_p, \qquad g_2 = 0$
Dissipation inequality:	$\int_\Omega [\langle \operatorname{dev} \operatorname{sym} \Sigma_E, \operatorname{sym} \dot{p} \rangle + \langle \operatorname{skew} \Sigma_E, \operatorname{skew} \dot{p} \rangle + g_1 \dot{\gamma}_p] \, dx \geq 0$
Dissipation function:	$\Delta(\dot{\Gamma}_p)$ is defined in (3.21)
Flow law in primal form:	$\Sigma_p \in \partial \Delta(\dot{\Gamma}_p)$
Flow law in dual form on $\mathcal{S}_1$:	$\begin{cases} \operatorname{sym} \dot{p} = \lambda \dfrac{\operatorname{dev} \operatorname{sym} \Sigma_E}{\|\operatorname{dev} \operatorname{sym} \Sigma_E\|}, & \dot{\gamma}_p = \lambda = \|\operatorname{sym} \dot{p}\| \\[2mm] \operatorname{skew} \dot{p} = 0, & \dot{\omega}_p = 0 \end{cases}$
Flow law in dual form on $\mathcal{S}_2$:	$\begin{cases} \operatorname{sym} \dot{p} = 0, & \dot{\gamma}_p = 0 \\[2mm] \operatorname{skew} \dot{p} = \lambda \dfrac{\operatorname{skew} \Sigma_E}{\|\operatorname{skew} \Sigma_E\|}, & \dot{\omega}_p = \lambda = \|\operatorname{skew} \dot{p}\| \end{cases}$
Flow law in dual form on $\mathcal{S}_3$:	$\begin{cases} \operatorname{sym} \dot{p} = \dfrac{2\lambda}{\sigma_0^2} (\|\operatorname{dev} \operatorname{sym} \Sigma_E\| + g_1) \dfrac{\operatorname{dev} \operatorname{sym} \Sigma_E}{\|\operatorname{dev} \operatorname{sym} \Sigma_E\|} \\[4mm] \operatorname{skew} \dot{p} = \dfrac{2\lambda}{\widehat{\sigma}_0^2} \operatorname{skew} \Sigma_E \\[4mm] \dot{\gamma}_p = \dfrac{2\lambda}{\sigma_0^2} (\|\operatorname{dev} \operatorname{sym} \Sigma_E\| + g_1) = \|\operatorname{sym} \dot{p}\| \\[4mm] \dot{\omega}_p = \dfrac{2\lambda}{\widehat{\sigma}_0^2} \|\operatorname{skew} \Sigma_E\| = \|\operatorname{skew} \dot{p}\| \\[4mm] \|\dot{p}\| = \sqrt{\dot{\gamma}_p^2 + \dot{\omega}_p^2}, \qquad \lambda = \dfrac{1}{2}\sqrt{\sigma_0^2 \dot{\gamma}_p^2 + \widehat{\sigma}_0^2 \dot{\omega}_p^2} \end{cases}$
KKT conditions:	$\lambda \geq 0, \qquad \phi(\Sigma_E, g_1, 0) \leq 0, \qquad \lambda \phi(\Sigma_E, g_1, 0) = 0$
Boundary conditions for p:	$p \times n = 0$ on $\Gamma, \quad (\operatorname{Curl} p) \times n = 0$ on $\partial\Omega \setminus \Gamma$
Function space for p:	$p(t, \cdot) \in \mathrm{H}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$

Table 3: The model with isotropic hardening only in the plastic strain-evolution (the case $\alpha_2 = 0$). Notice that the boundary conditions on p necessitates at least $p \in \mathrm{H}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$, which is not guaranteed looking at the free-energy and the dissipation function. However, this will be obtained from a new Korn's type inequality for incompatible tensor fields derived by Neff et al. in [86, 87, 88, 89].

5.3. Existence result in the no-spin-hardening case

As in Section 4.3, we write (5.3) as the variational inequality of the second kind: find
$w = (u, p, \gamma_p, \omega_p) \in \mathsf{H}^1(0, T; \mathsf{Z})$ such that $w(0) = 0$ and $\dot{w}(t) \in \mathsf{W}$ for a.e. $t \in [0, T]$

$$\boldsymbol{a}(\dot{w}, z - w) + j(z) - j(\dot{w}) \geq \langle \ell, z - \dot{w} \rangle \text{ for every } z \in \mathsf{Z} \text{ and for a.e. } t \in [0, T], \tag{5.4}$$

where Z is a suitable Hilbert space and W is a closed, convex subset of Z to be constructed and

$$\boldsymbol{a}(w,z) = \int_\Omega \Big[\langle \mathbb{C}_{\mathrm{iso}}(\mathrm{sym}\,\nabla u - \mathrm{sym}\,p), \mathrm{sym}\,\nabla v - \mathrm{sym}\,q)\rangle$$

$$+ \mu\,L_c^2 \langle \mathrm{Curl}\,p, \mathrm{Curl}\,q\rangle + \mu\,\alpha_1 \gamma_p\,\eta \Big]\,dx\,, \tag{5.5}$$

$$j(z) = \int_\Omega \Delta(q,\eta,\beta)\,dx\,, \tag{5.6}$$

$$\langle \ell, z\rangle = \int_\Omega f\,v\,dx\,, \tag{5.7}$$

for $w = (u, p, \gamma_p, \omega_p)$ and $z = (v, q, \eta, \beta)$ in W.

Similarly to Section 4.3, the main challenge here is to construct the space Z and the subset W such that the bilinear form $\boldsymbol{a}$ is coercive in W.

First of all, notice that since the bilinear form $\boldsymbol{a}$ does not contain explicitly the variable β, it is impossible to derive the coercivity of the bilinear form in any normed space in all the variables v, q, η and β. Therefore, the new solution strategy here for the existence result is to first find u, p and γ_p, and construct ω_p a posteriori. To this end, we define

$$\Delta_0(q, \eta) := \begin{cases} D(\|\mathrm{sym}\,q\|, \|\mathrm{skew}\,q\|) & \text{if } \|\mathrm{sym}\,q\| \leq \eta\,, \\ \\ \infty & \text{otherwise}\,, \end{cases} \tag{5.8}$$

where we recall that

$$D(s,t) := \sqrt{\sigma_0^2\,s^2 + \widehat{\sigma}_0^2\,t^2}\,. \tag{5.9}$$

We then reformulate the problem as follows: find $w = (u, p, \gamma_p) \in \mathsf{H}^1(0, T; Z)$ such that $w(0) = 0$, $\dot{w}(t) \in W$ for a.e. $t \in [0, T]$

$$\boldsymbol{a}(\dot{w}, z - w) + j_0(z) - j_0(\dot{w}) \geq \langle \ell, z - \dot{w}\rangle \text{ for every } z = (v, q, \eta) \in W \text{ and for a.e. } t \in [0, T]\,, \tag{5.10}$$

where we let

$$j_0(z) := \begin{cases} \displaystyle\int_\Omega \Delta_0(q, \eta)\,dx & \text{if } z = (v, q, \eta) \in W \\ \\ \infty & \text{otherwise}\,, \end{cases} \tag{5.11}$$

$$\mathsf{Z} = \mathsf{V} \times \mathsf{Q} \times \Lambda\,, \tag{5.12}$$

$$\mathsf{W} = \big\{ z = (v, q, \eta) \in \mathsf{Z} \mid \|\mathrm{sym}\,q\| \leq \eta \text{ a.e. in } \Omega \big\}\,, \tag{5.13}$$

$$\mathsf{V} = \mathsf{H}_0^1(\Omega, \Gamma_D, \mathbb{R}^3) = \big\{ v \in \mathsf{H}^1(\Omega, \mathbb{R}^3) \mid v_{|\Gamma_D} = 0 \big\}\,,$$

$$\mathsf{Q} = \mathsf{H}_0(\mathrm{Curl};\Omega, \Gamma, \mathfrak{sl}(3)) \quad \text{defined in (2.7)}\,,$$

$$\Lambda = L^2(\Omega)\,,$$

equipped with the norms

$$\|v\|_V := \|\nabla v\|_{L^2}\,, \qquad \|q\|_Q := \|q\|_{\mathsf{H}(\mathrm{Curl};\Omega)}\,,$$

$$\|z\|_Z^2 := \|v\|_V^2 + \|q\|_Q^2 + \|\eta\|_{L^2}^2 \quad \text{for } z = (v, q, \eta, \beta) \in \mathsf{Z}\,. \tag{5.14}$$

Now, for the existence of a solution to the problem (5.10) following Hahn-Reddy [55, Theorem 7.3], we only need to check that the bilinear form $\boldsymbol{a}$ is coercive in W. Following the coercivity inequality in (4.12), we immediately get a positive constant $C = C(m_0, \mu, \alpha_1, L_c, \Omega) > 0$ such that

$$a(z, z) \geq C\big[\|v\|_V^2 + \|\operatorname{sym} q\|^2 + \|\operatorname{Curl} q\|_2^2 + \|\eta\|_2^2\big]\,.$$

But this estimate is not enough to establish coercivity. Indeed, the skew-symmetric (spin) part $\operatorname{skew} q$ of q is not controlled locally.

Motivated by the well-posedness question for precursors to this model [83, 27], Neff et al. [86, 87, 88, 89], derived a new inequality extending Korn's first inequality to incompatible tensor fields, namely there exists a constant $C(\Omega) > 0$ such that

$$\forall\, p \in \mathsf{H}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3}), \quad p \times n|_\Gamma = 0: \tag{5.15}$$

$$\underbrace{\|p\|_{L^2(\Omega)}}_{\text{plastic distortion}} \leq C(\Omega) \Big(\underbrace{\|\operatorname{sym} p\|_{L^2(\Omega)}}_{\text{plastic strain}} + \underbrace{\|\operatorname{Curl} p\|_{L^2(\Omega)}}_{\text{dislocation density}} \Big)\,.$$

This shows that if we consider the closure $\mathsf{H}_{\text{sym}}\,(\operatorname{Curl},\, \Omega, \Gamma; \mathfrak{sl}(3))$ of the linear subspace

$$\{q \in C^\infty(\overline{\Omega}, \mathbb{R}^{3\times 3}) \mid \operatorname{tr} q = 0,\ q \times n = 0 \text{ on } \Gamma\}$$

in the norm

$$\|q\|_{\text{sym, curl}}^2 := \|\operatorname{sym} q\|_{L^2}^2 + \|\operatorname{Curl} q\|_{L^2}^2\,, \tag{5.16}$$

then we have the decisive identity

$$\mathsf{H}_{\text{sym}}\,(\operatorname{Curl},\, \Omega, \Gamma; \mathfrak{sl}(3)) \equiv \mathsf{H}_0(\operatorname{Curl},\, \Omega, \Gamma; \mathfrak{sl}(3))$$

with equivalence of norms. Therefore, we have the coercivity inequality

$$\boldsymbol{a}(z, z) \geq C\big[\|v\|_V^2 + \|q\|_Q^2 + \|\eta\|_2^2\big] = \|z\|_Z^2 \qquad \forall z \in \mathsf{W}\,, \tag{5.17}$$

from which we obtain the existence of a solution $(u, p, \gamma_p) \in \mathsf{W}$ to the problem (5.10). Now setting a posteriori

$$\omega_p(t, x) := \int_0^t \|\operatorname{skew} \dot{p}(s, x)\|\, ds\,, \tag{5.18}$$

it follows that $(u, p, \gamma_p, \omega_p)$ is a solution to the original problem (5.4).

Remark 5.1 Notice again that isotropic hardening in the spin-evolution is not necessary for existence of a solution to the problem and it is not connected to the uniqueness question either. In fact, arguing as in Section 4.4 we get the inequality (4.21) with $\alpha_2 = 0$, from which and from the flow law in dual form on each $\mathcal{S}_k$ of the yield surface, we deduce the uniqueness of u, σ, p and γ_p while the uniqueness of ω_p follows from (5.18) and from the uniqueness of p. Therefore, the strong solution is unique also in the case where there is no isotropic hardening in the spin-evolution.

5.4. Is it possible to accommodate the special case $\widehat{\sigma}_0 = 0$ in our model?

In Gurtin's visco-plastic model [41] it is possible to consider $\widehat{\sigma}_0 = 0$. In our setting, this case corresponds to the dissipation function

$$\Delta(q, \eta, \beta) := \begin{cases} \sigma_0 \, \|\operatorname{sym} q\| & \text{if } \|\operatorname{sym} q\| \leq \eta \quad \text{and} \quad \|\operatorname{skew} q\| \leq \beta \\ \infty & \text{otherwise} \end{cases} \tag{5.19}$$

and the elastic region

$$\mathcal{E} := \left\{ \Sigma_p = (\Sigma_E, g_1, g_2) \mid \|\operatorname{dev} \operatorname{sym} \Sigma_E\| - \sigma_0 + g_1 \leq 0 \text{ and } \|\operatorname{skew} \Sigma_E\| + g_2 \leq 0 \right\}. \tag{5.20}$$

The flow law in dual form is given in Table 4 below.

Free energy:	$\frac{1}{2} \langle \mathbb{C}_{\mathrm{iso}} \varepsilon^e, \varepsilon^e \rangle + \frac{1}{2} \mu \, L_c^2 \, \|\operatorname{Curl} p\|^2 + \frac{1}{2} \mu \, \alpha_1 \, |\gamma_p|^2 + \frac{1}{2} \mu \, \alpha_2 |\omega_p|^2$
Elastic region:	$\mathcal{E} := \left\{ (\Sigma_E, g_1, g_2) \mid \|\operatorname{dev} \operatorname{sym} \Sigma_E\| - \sigma_0 + g_1 \leq 0 \text{ and } \|\operatorname{skew} \Sigma_E\| + g_2 \leq 0 \right\}$
Yield surface:	$\partial \mathcal{E} = \mathcal{S}_1 \cup \mathcal{S}_2$
where	$\mathcal{S}_1 := \left\{ (\Sigma_E, g_1, g_2) \mid \|\operatorname{dev} \operatorname{sym} \Sigma_E\| - \sigma_0 + g_1 = 0 \right\}$
	$\mathcal{S}_2 := \left\{ (\Sigma_E, g_1, g_2) \mid \|\operatorname{skew} \Sigma_E\| + g_2 = 0 \right\}$
Dissipation function:	$\Delta(q, \eta, \beta) := \begin{cases} \sigma_0 \, \|\operatorname{sym} q\| & \text{if } \|\operatorname{sym} q\| \leq \eta \quad \text{and} \quad \|\operatorname{skew} q\| \leq \beta \\ \infty & \text{otherwise} \end{cases}$
Flow law in dual form:	$\begin{cases} \operatorname{sym} \dot{p} = \lambda \dfrac{\operatorname{dev} \operatorname{sym} \Sigma_E}{\|\operatorname{dev} \operatorname{sym} \Sigma_E\|}, \quad \operatorname{skew} \dot{p} = 0, \quad \dot{\gamma}_p = \lambda, \quad \dot{\omega}_p = 0 \quad \text{on } \mathcal{S}_1 \\[2ex] \operatorname{sym} \dot{p} = 0, \quad \operatorname{skew} \dot{p} = \lambda \dfrac{\operatorname{skew} \Sigma_E}{\|\operatorname{skew} \Sigma_E\|}, \qquad \dot{\gamma}_p = 0, \quad \dot{\omega}_p = \lambda \quad \text{on } \mathcal{S}_2 \end{cases}$

Table 4: The flow rule in dual form in the case $\widehat{\sigma}_0 = 0$ and $\alpha_2 > 0$.

Table 4 illustrates why both initial yield stresses σ_0 and $\widehat{\sigma}_0$ have to be strictly positive. In fact, since g_2 may be zero initially, the elastic domain $\mathcal{E}$ in (5.20) may not have non-empty interior. Therefore, this forbids the use of $\widehat{\sigma}_0 = 0$. In the visco-plastic setting $\widehat{\sigma}_0 = 0$ may be accommodated.

6. The limit case of vanishing characteristic length scale $L_c \to 0$

In the limit case $L_c \to 0$, looking at the flow rule in its dual formulation, we first observe that the thermodynamic driving stress $\Sigma_E \in \mathbb{R}^{3\times 3}$ reduces to the symmetric Cauchy stress tensor $\sigma \in \operatorname{Sym}(3)$ and we see clearly that we do not have the branch $\mathcal{S}_2$ and moreover,

$$\text{on } \mathcal{S}_1 : \quad \begin{cases} \text{sym}\,\dot{p} \;=\; \lambda\,\dfrac{\text{dev}\,\sigma}{\|\text{dev}\,\sigma\|}, \quad & \dot{\gamma}_p = \lambda = \|\text{sym}\,\dot{p}\|, \\[2ex] \text{skew}\,\dot{p} = 0, & \dot{\omega}_p = 0\,, \end{cases} \tag{6.1}$$

while on $\mathcal{S}_3$ we get from the rate-explicit dual formulation

$$\begin{cases} \text{sym}\,\dot{p} = \dfrac{2\lambda}{\sigma_0^2}(\|\text{dev}\,\text{sym}\,\Sigma_E\| + g_1)\dfrac{\text{dev}\,\text{sym}\,\Sigma_E}{\|\text{dev}\,\text{sym}\,\Sigma_E\|}\,, \quad & \dot{\gamma}_p = \dfrac{2\lambda}{\sigma_0^2}(\|\text{dev}\,\text{sym}\,\Sigma_E\| + g_1) = \|\text{sym}\,\dot{p}\| \\[3ex] \text{skew}\,\dot{p} = \dfrac{2\lambda}{\widehat{\sigma}_0^2}(\|\text{skew}\,\Sigma_E\| + g_2)\dfrac{\text{skew}\,\Sigma_E}{\|\text{skew}\,\Sigma_E\|}\,, & \dot{\omega}_p = \dfrac{2\lambda}{\widehat{\sigma}_0^2}(\|\text{skew}\,\Sigma_E\| + g_2) = \|\text{skew}\,\dot{p}\| \end{cases}$$

in the case $L_c > 0$ and $\alpha_2 > 0$ and

$$\begin{cases} \text{sym}\,\dot{p} = \dfrac{2\lambda}{\sigma_0^2}(\|\text{dev}\,\text{sym}\,\Sigma_E\| + g_1)\dfrac{\text{dev}\,\text{sym}\,\Sigma_E}{\|\text{dev}\,\text{sym}\,\Sigma_E\|}\,, \quad & \dot{\gamma}_p = \dfrac{2\lambda}{\sigma_0^2}(\|\text{dev}\,\text{sym}\,\Sigma_E\| + g_1) = \|\text{sym}\,\dot{p}\| \\[3ex] \text{skew}\,\dot{p} = \dfrac{2\lambda}{\widehat{\sigma}_0^2}\,\text{skew}\,\Sigma_E\,, & \dot{\omega}_p = \dfrac{2\lambda}{\widehat{\sigma}_0^2}\|\text{skew}\,\Sigma_E\| = \|\text{skew}\,\dot{p}\| \end{cases}$$

in the case $L_c > 0$ and $\alpha_2 = 0$ that altogether

$$\text{on } \mathcal{S}_3 : \quad \begin{cases} \text{sym}\,\dot{p} \;=\; \dfrac{2\lambda}{\sigma_0}\dfrac{\text{dev}\,\sigma}{\|\text{dev}\,\sigma\|}, \quad & \dot{\gamma}_p = \dfrac{2\lambda}{\sigma_0} = \|\text{sym}\,\dot{p}\|, \\[2ex] \text{skew}\,\dot{p} = 0, & \dot{\omega}_p = 0\,. \end{cases} \tag{6.2}$$

Therefore, we obtain for $L_c \to 0$ that all driving stress-tensor quantities are symmetric such that, if $p(0) \in \text{Sym}(3)$, then we will have $p(t) \in \text{Sym}(3)$ along the plastic evolution. In that case, our new model turns into

$$\dot{\varepsilon}_p = \widehat{\lambda}\,\dfrac{\text{dev}\,\sigma}{\|\text{dev}\,\sigma\|}\,, \qquad \dot{\gamma}_p = \widehat{\lambda} = \|\dot{\varepsilon}_p\|\,, \tag{6.3}$$

which is the dual formulation of the flow rule for classical plasticity with isotropic hardening based only on the accumulated equivalent plastic strain $\gamma_p = \int_0^t \|\dot{\varepsilon}_p\|\, ds$.

For us it is interesting to remark that the evolution of plastic spin in our model is related solely to the energetic length scale $L_c > 0$.

7. Conclusions and outlook

¿From a modelling perspective, it is not difficult to extend the present model to visco-plasticity. However, the well-posedness result (which we expect to hold) needs to be derived along different methods. Moreover, it would be interesting to treat the dynamic case. Both questions are subject of ongoing work.

Since we did not establish unqualified uniqueness in our model (it hinges on the additional regularity Curl Curl $p \in L^2(\Omega, \mathbb{R}^{3\times 3})$) it will also be interesting to establish higher regularity provided the data are regular. It remains open whether we really could have non-uniqueness of the weak solutions if regularity is missing. Is the dislocation energy contribution Curl $p \in L^2(\Omega, \mathbb{R}^{3\times 3})$ strong enough to prevent non-uniqueness? The question we have to answer is, what least amount of hardening will lead to existence and uniqueness in rate-independent gradient plasticity?

We expect furthermore that a computational implementation suggests itself along the lines of [90]. Attendant to these research perspectives, one should look at simple settings of boundary value problems like anti-plane shear to gain more insight in the response of the model and the new features offered by incorporating plastic spin.

Finally, a major challenge from the mathematical point of view is the replacement of the defect energy in $(3.5)_2$ by a more physically realistic term $\mu\, L_c \|\text{Curl } p\|$. Such a defect energy was proposed in [96] in the context of single crystal gradient plasticity and is summarized in [55, p.92]. The successful mathematical treatment of such a model needs fundamentally new ideas.

8. Appendix

8.1. Fenchel-Legendre transformation and admissible stresses

The following result was used in (3.34).

$$\sup_{s\geq 0,\, t\geq 0}\left\{As + Bt - \sqrt{s^2 + t^2}\right\} = \begin{cases} 0 & \text{if } \begin{cases} A \leq 1 & \text{if } \quad B \leq 0 \\ B \leq 1 & \text{if } \quad A \leq 0 \\ A^2 + B^2 \leq 1 & \text{if } \begin{cases} A \geq 0 \\ B \geq 0 \end{cases} \end{cases} \\ \\ \infty & \text{otherwise}. \end{cases} \tag{8.1}$$

Notice that the function $f : \mathbb{R}^2 \to \overline{\overline{\mathbb{R}}}$ defined by

$$f(A, B) := \sup_{s\geq 0,\, t\geq 0}\left\{As + Bt - \sqrt{s^2 + t^2}\right\}$$

is the Fenchel-Legengre conjugate of the function

$$g(s, t) := \begin{cases} \sqrt{s^2 + t^2} & \text{if } s \geq 0,\, t \geq 0, \\ \\ \infty & \text{otherwise}. \end{cases}$$

Let us show that (8.1) holds.

- If $B \leq 0$, then

$$\sup_{s\geq 0,\, t\geq 0}\left\{As + Bt - \sqrt{s^2 + t^2}\right\} = \sup_{t\geq 0}\{t(A - 1)\} = \begin{cases} 0 & \text{if } A \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

- Similarly, if $A \leq 0$, then

$$\sup_{s \geq 0, t \geq 0} \left\{ As + Bt - \sqrt{s^2 + t^2} \right\} = \begin{cases} 0 & \text{if } B \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

- If $A \geq 0$ and $B \geq 0$, then

$$\sup_{s \geq 0, t \geq 0} \left\{ As + Bt - \sqrt{s^2 + t^2} \right\} = \sup_{r \geq 0} \{ r[(A^2 + B^2)^{1/2} - 1] \} = \begin{cases} 0 & \text{if } A^2 + B^2 \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

As noticed in Section 3.3.1, we could consider a more general dissipation function corresponding to the function

$$\widehat{D}(s,t) := r_1\, s + r_2\, t + \sqrt{\sigma_0^2\, s^2 + \widehat{\sigma}_0^2\, t^2} \qquad \text{with } r_1, r_2 \geq 0. \tag{8.2}$$

For such a choice, we get following the calculation done to find the elastic domain in Section 3.3.2 that

$$I_{\mathcal{E}}(\Sigma_p) = \sup_{s \geq 0, t \geq 0} \left\{ \frac{A}{\sigma_0} s + \frac{B}{\widehat{\sigma}_0} t - \sqrt{s^2 + t^2} \right\} = f\left(\frac{A}{\sigma_0}, \frac{B}{\widehat{\sigma}_0} \right) \tag{8.3}$$

with

$$A := \|\text{dev sym}\, \Sigma_E\| - r_1 + g_1 \quad \text{and} \quad B := \|\text{skew}\, \Sigma_E\| - r_2 + g_2. \tag{8.4}$$

Therefore, in this case the set of admissible stresses is

$$\mathcal{E} = \left\{ (\Sigma_E, g_1, g_2) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^- \times \mathbb{R}^- \mid (\|\text{dev sym}\, \Sigma_E\| + g_1 + r_1, \|\text{skew}\, \Sigma_E\| + g_2 + r_2) \in \mathcal{K} \right\}, \tag{8.5}$$

with the set $\mathcal{K}$ defined in (3.35). The set $\mathcal{E}$ here corresponds to a dilation of the set in (3.36).

8.2. Visco-plastic regularizaion of our model

In the table below we propose a Norton-Hoff visco-plastic regularization of our model which is currently a subject of an ongoing work.

Acknowledgements:

Additive split of distortion:	$\nabla u = e + p, \quad \varepsilon^e = \mathrm{sym}\, e, \quad \varepsilon^p = \mathrm{sym}\, p$
Equilibrium:	$\mathrm{Div}\,\sigma + f = 0$ with $\sigma = \mathbb{C}_{\mathrm{iso}}\varepsilon^e$
Free energy:	$\frac{1}{2}\langle \mathbb{C}_{\mathrm{iso}}\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\mu\,L_c^2\,\|\mathrm{Curl}\,p\|^2 + \frac{1}{2}\mu\,\alpha_1\,|\gamma_p|^2 + \frac{1}{2}\mu\,\alpha_2\,|w_p|^2$
Yield condition:	$\phi(\Sigma_p) = 0$ with ϕ given in (4.1)
where	$\Sigma_p = (\Sigma_E, g_1, g_2), \quad \Sigma_E := \sigma + \Sigma_{\mathrm{curl}}^{\mathrm{lin}}, \quad \Sigma_{\mathrm{curl}}^{\mathrm{lin}} = -\mu\,L_c^2\,\mathrm{Curl}\,\mathrm{Curl}\,p$ $g_1 = -\mu\,\alpha_1\gamma_p, \quad g_2 = -\mu\,\alpha_2\,w_p$
Dissipation inequality:	$\int_\Omega [\langle \mathrm{dev}\,\mathrm{sym}\,\Sigma_E, \mathrm{sym}\,\dot{p}\rangle + \langle \mathrm{skew}\,\Sigma_E, \mathrm{skew}\,\dot{p}\rangle + g_1\,\dot{\gamma}_p + g_2\,\dot{w}_p]\,dx \geq 0$
Dissipation function:	$\Delta(\dot{\Gamma}_p)$ is defined in (3.21)
Flow law:	$\begin{cases} \mathrm{sym}\,\dot{p} &= \frac{1}{\rho}[\|\mathrm{dev}\,\mathrm{sym}\,\Sigma_E\| - \sigma_0 + g_1]_+^n\,\frac{\mathrm{dev}\,\mathrm{sym}\,\Sigma_E}{\|\mathrm{dev}\,\mathrm{sym}\,\Sigma_E\|} \\[2mm] \mathrm{skew}\,\dot{p} &= \frac{1}{\rho}[\|\mathrm{skew}\,\Sigma_E\| - \widehat{\sigma}_0 + g_2]_+^m\,\frac{\mathrm{skew}\,\Sigma_E}{\|\mathrm{skew}\,\Sigma_E\|} \\[2mm] \dot{\gamma}_p &= \frac{1}{\rho}[\|\mathrm{dev}\,\mathrm{sym}\,\Sigma_E\| - \sigma_0 + g_1]_+^n = \|\mathrm{sym}\,\dot{p}\| \\[2mm] \dot{\omega}_p &= \frac{1}{\rho}[\|\mathrm{skew}\,\Sigma_E\| - \widehat{\sigma}_0 + g_2]_+^m = \|\mathrm{skew}\,\dot{p}\| \end{cases}$
where	$[a]_+ = \max\{a,0\}, \quad \rho > 0$ a viscosity constant, $\quad n,\,m \in \mathbb{N}$ viscosity exponents
Boundary conditions for p:	$p \times n = 0$ on Γ, $\quad (\mathrm{Curl}\,p) \times n = 0$ on $\partial\Omega \setminus \Gamma$
Function space for p:	$p(t,\cdot) \in \mathrm{H}(\mathrm{Curl};\,\Omega,\,\mathbb{R}^{3\times3})$

Table 5: A Norton-Hoff-type visco-plastic regularization of our model in the case $\alpha_2 > 0$.

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