A Korn's inequality for incompatible tensor fields

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Dedicated to Professor Rolf Leis on the occasion of his 80th birthday.

We prove a Korn-type inequality for bounded Lipschitz domains Ω in $\mathbb{R}^3$ and non-symmetric square integrable tensor fields $P : \Omega \to \mathbb{R}^{3\times3}$ having square integrable rotation $\operatorname{Curl} P : \Omega \to \mathbb{R}^{3\times3}$. For skew-symmetric P or compatible $P = \nabla v$ our estimate reduces to non-standard variants of Poincaré's or Korn's first inequality, respectively, for which our new estimate can be viewed as a common generalized version.

1 Results

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ. Moreover, let Γ_t be a relatively open subset of Γ. We prove the following Korn-type inequality: There exists a constant $c > 0$ such that for all $P \in C^\infty(\mathbb{R}^3, \mathbb{R}^{3\times3})$

$$c \int_\Omega |P|^2 \leq \int_\Omega |\operatorname{sym} P|^2 + \sum_{n=1}^{3} \int_\Omega |\operatorname{curl} P_n|^2 + \sum_{n=1}^{3} \int_{\Gamma_t} |\nu \times P_n|^2.$$

Here, ν denotes the outward unit normal at Γ and the tensor field P consists of the rows P_n^T, where P_n are vector fields in $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Of course, this estimate holds true in the appropriate Hilbert space setting. Let $H(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$ be the Hilbert space of all tensor fields having rows in $H(\operatorname{curl}; \Omega, \mathbb{R}^3)$, the well known Sobolev space generated by curl.

Theorem 1.1 *There exists a constant $c > 0$ such that for all $P \in H(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$ the estimate*

$$c||P||_{L^2(\Omega,\mathbb{R}^{3\times3})} \leq ||\operatorname{sym} P||_{L^2(\Omega,\mathbb{R}^{3\times3})} + \sum_{n=1}^{3} ||\operatorname{curl} P_n||_{L^2(\Omega,\mathbb{R}^3)} + \sum_{n=1}^{3} ||\tau_t P_n||_{H^{-1/2}(\Gamma_t)}$$

holds true, where τ_t denotes the restricted tangential trace.

Our proofs rely on three essential tools, namely

1. Maxwell's estimate (Poincaré-type estimate for curl and div);
2. Helmholtz' decomposition;
3. Korn's first inequality.

There are immediate consequences of Theorem 1.1.

Corollary 1.2 *There exists a constant $c > 0$ such that*

$$c||P||_{L^2(\Omega,\mathbb{R}^{3\times3})} \leq ||\operatorname{sym} P||_{L^2(\Omega,\mathbb{R}^{3\times3})} + \sum_{n=1}^{3} ||\operatorname{curl} P_n||_{L^2(\Omega,\mathbb{R}^3)}$$

holds for all $P \in H(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$ with vanishing restricted tangential trace on Γ_t.

Corollary 1.3 *(generalized Korn's first inequality) There exists a constant $c > 0$ such that*

$$c||\nabla v||_{L^2(\Omega,\mathbb{R}^{3\times3})} \leq ||\operatorname{sym} \nabla v||_{L^2(\Omega,\mathbb{R}^{3\times3})}$$

holds for all vector fields $v \in H^1(\Omega, \mathbb{R}^3)$ vanishing on Γ_t or having components ∇v_n normal at Γ_t.

Remark 1.4 (generalized Poincaré's inequality) Since vector fields can be identified with skew-symmetric tensor fields and Curl bounds (pointwise) the full gradient of those, we obtain by Theorem 1.1 and Corollary 1.2 new variants of Poincaré's inequality.

We want to note that similar results hold in $\mathbb{R}^N$ and in the Banach space setting of L^p-spaces.

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2 Application

In the theory of extended continuum mechanics we encounter the micromorphic approach. A subvariant of this model can be written in the form of a minimization problem for two fields, i.e., the classical displacement $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ and the micromorphic tensor field $P : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^{3\times3}$. The additional field may be needed in the description of foams and bones [1–4]. The problem is to find the pair (u, P) such that

$$\int_\Omega \mu|\operatorname{sym}(\nabla u - P)|^2 + \frac{\lambda}{2}|\operatorname{tr}(\nabla u - P)|^2 - f\cdot u + h^+\big(\mu|\operatorname{sym}P|^2 + \frac{\lambda}{2}|\operatorname{tr}P|^2\big) + \mu L_c^2|\operatorname{Curl}P|^2 \longrightarrow \min$$

subject to the boundary conditions of place $u|_{\Gamma_t} = 0$ and $\nu \times P|_{\Gamma_t} = 0$. Here, the problem is driven by the body force f and $L_c > 0$ has dimensions of length, $h^+ > 0$ is a non-dimensional factor and μ, λ are the Lamé-constants of the material. With our estimate at hand one can show that the unique solution satisfies $u \in H^1(\Omega, \mathbb{R}^3)$ and $P \in H(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$. In order that the model is invariant with respect to superposed infinitesimal rigid rotations, i.e., $(\nabla u, P) \mapsto (\nabla u + A, P + A)$ for constant $A \in \mathfrak{so}(3)$, the symmetric local contribution $\operatorname{sym}P$ is mandatory.

Provided that $h^+ = 1$, this formulation is a relaxed formulation of linear elasticity, since the stored energy will always be less than the stored energy for the corresponding linear elastic formulation: Just take $P = \nabla u$, where u is the classical solution. This remains true even in the formal limit $L_c \to \infty$.

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