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# A model describing small elastic deformations and Korn's inequality with nonconstant coefficients

*This contribution is concerned with the formulation and mathematical investigation of a model for small elastic deformations which arises from multiplicative theories of elasto-plasticity. In a natural way it leads to a linear elliptic system with nonconstant coefficients for the deformation u. In contrast to infinitesimal plasticity the model should be valid for both large plastic deformations $F_p$ and large deformation gradients F. The arising linear partial differential system is proved to have unique solutions by means of a generalized Korn's inequality.*

## 1. Motivation

In the nonlinear theory of elasto-plasticity at large deformation gradients it is often assumed that the deformation gradient $F = \nabla u$ splits multiplicatively into an elastic and plastic part $\nabla u(x) = F(x) = F_e(x) \cdot F_p(x)$, $F_e, F_p \in GL(3, \mathbb{R})$ where $F_e, F_p$ are explicitly understood to be incompatible configurations, i.e $F_e, F_p \neq \nabla\Psi$ for any $\Psi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$. In our context we assume that this decomposition is uniquely defined up to a rigid rotation. This ansatz is micromechanically motivated by the kinematics of single crystals where dislocations move along fixed slip systems through the crystal lattice. The source for the incompatibility are those dislocations which did not completely transverse the crystal and consequently give rise to an inhomogeneous plastic deformation. Therefore it seems reasonable to introduce the deviation of the plastic intermediate configuration $F_p$ from compatibility as a kind of plastic **dislocation density**. This deviation should be related somehow to the quantity $Rot F_p$ and indeed later on we see the important role which is played by $Rot F_p$, see [4] for more on this subject and for applications of this theory in the engineering field look e.g at [2,3].

## 2. Metal Plasticity

It is known that any homogeneous, isotropic and material objective energy with stress free reference configuration $\mathbb{1}$ admits the representation

$$\tilde{W}(F) = \lambda\, \|F^T F - \mathbb{1}\|^2 + \mu\, tr(F^T F - \mathbb{1})^2 + o(\|F^T F - \mathbb{1}\|^2) \tag{1}$$

near $\mathbb{1}$. Here $\lambda, \mu > 0$ denote the Lamé constants. When dealing with metal-plasticity it is observed that elastic deformations remain small in the sense that $\|F_e^T F_e - \mathbb{1}\|$ remains pointwise small. Accordingly taking $\tilde{W}$ and inserting $F_e$ instead of $F$ and skipping the higher order term $o(\|F_e^T F_e - \mathbb{1}\|^2)$ the following St. Venant-Kirchhoff ansatz for a hyperelastic free energy should be a reasonable first choice:

$$\hat{W} = \hat{W}(F_e) = \lambda\, \|F_e^T F_e - \mathbb{1}\|^2 + \mu\, tr(F_e^T F_e - \mathbb{1})^2. \tag{2}$$

However, $\hat{W}$ would still lead to a problem which is neither linear in $F$ nor elliptic. Therefore invoking the smallness of $\|F_e^T F_e - \mathbb{1}\|$ again we see with the aid of the polar decomposition that $F_e$ is approximately a rotation $R_e$. If we set $F_e = (F_e - R_e) + R_e$, insert this formula into the free energy $\hat{W}$ and cancel terms which are of second order in $(F_e - R_e)$ we are left with the following elastic energy:

$$W(F, F_p, R_e) = \lambda\, \|R_e^T F\, F_p^{-1} + F_p^{-T} F^T R_e - 2 \cdot \mathbb{1}\|^2 + \mu\, tr(R_e^T F\, F_p^{-1} + F_p^{-T} F^T R_e - 2 \cdot \mathbb{1})^2. \tag{3}$$

Note that $W$ is quadratic with respect to $F$ if $F_p, R_e$ are assumed to be known.. The new energy $W$ is still material objective since $Q \cdot F = Q \cdot F_e \cdot F_p$ implies $R_e(Q \cdot F_e) = Q \cdot R_e(F_e)$ and $W(Q \cdot F, F_p, Q \cdot R_e) = W(F, F_p, R_e)\ \forall\ Q \in O(3)$. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain with boundary $\partial\Omega$. In the absence of body forces and in the quasistatic setting the problem to be solved is: find the deformation $u : [0, T] \times \Omega \mapsto \mathbb{R}^3$ and the plastic deformation gradient $F_p : [0, T] \times \Omega \mapsto GL(3, \mathbb{R})$ such that

$$div\, D_F W(F(x,t), F_p(x,t), R_e(x,t)) = 0 \quad x \in \Omega \tag{4}$$

$$\frac{d}{dt} F_p^{-1} = f(F, F_p^{-1}) \tag{5}$$

$$\begin{aligned} R_e(x,t) &= polar(F_e(x,t)) && (6)\\ F(x,t) &= F_e(x,t)\cdot F_p(x,t) && (7)\\ F_p^{-1}(x,0) &= F_{p_0}^{-1}(x) && (8)\\ u_{|\partial\Omega}(x,t) &= g(x,t) && (9)\end{aligned}$$

where $f : \mathbb{M}^{3\times3}\times\mathbb{M}^{3\times3}\mapsto\mathbb{M}^{3\times3}$ is some function governing the plastic evolution, $g$ is the given Dirichlet boundary data and $F_{p_0}^{-1}$ is the initial condition on the plastic flow. Here $polar : GL(3,\mathbb{R})\mapsto O(3)$ is the function which gives the unique rotation of its argument according to the polar decomposition. Observe that the complete system is still nonlinear in $F$ altogether due to the appearance of $polar : GL(3,\mathbb{R})\mapsto O(3)$. Some simple computations reveal that the above equilibrium system is a linear elliptic system with nonconstant coefficients at fixed values $F_p, R_e$ in contrast to the elliptic system with constant coefficients in infinitesimal plasticity. Note that $R_e$ represents in a natural way deformation induced anisotropy. It is natural to ask whether at fixed time $t_0$ the equilibrium equation $div\, D_F W(F, F_p, R_e) = 0$ has a unique solution if the data $g, F_p, R_e$ at $t_0$ are known. The answer is given in

Theorem 1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded smooth domain with smooth boundary and let $F_p, R_e\in C^2(\overline{\Omega}, GL(3,\mathbb{R}))$. Moreover assume that $g\in H^1(\Omega,\mathbb{R}^3)$. Then*

$$\begin{aligned} div\, D_F W(F,F_p,R_e) &= 0 \quad x\in\Omega && (10)\\ u_{|\partial\Omega} &= g && (11)\end{aligned}$$

*admits a unique solution $u\in H^1(\Omega,\mathbb{R}^3)$.*

Proof. The proof uses the key idea to interprete the equilibrium equation as the Euler-Lagrange equation of the functional $I : H^1(\Omega,\mathbb{R}^3)\times C^2(\overline{\Omega}, GL(3,\mathbb{R}))\times C^2(\overline{\Omega}, GL(3,\mathbb{R}))\mapsto\mathbb{R}$ with $I(u,F_p,R_e) := \int_\Omega W(\nabla u, F_p^{-1}, R_e)\,dx$. Evaluating the second derivative of $I$ with respect to $u$ we have the following estimate

$$D_u^2 I(u,F_p,R_e).(\phi,\phi)\geq 2\lambda\int_\Omega \|F_p^{-T}\nabla\phi^T R_e + R_e^T\nabla\phi F_p^{-1}\|^2\,dx. \tag{12}$$

In [1] it is shown by proving a generalized Korn's inequality that there exists some positive constant $c^+>0$ such that forall $\phi\in H_0^1(\Omega,\mathbb{R}^3)$ we have

$$\int_\Omega \|F_p^{-T}\nabla\phi^T R_e + R_e^T\nabla\phi F_p^{-1}\|^2\,dx\geq c^+\cdot\|\phi\|^2_{H^1(\Omega)} \tag{13}$$

which implies the strict convexity of $I$ with respect to $u$. By the direct methods of the calculus of variations it is clear that there exists a unique minimizer of $I$ over the space $H^1(\Omega)$ together with the boundary condition. In the prove of this assertion a prominent role is played by the quantity $Rot F_p$ which to our opinion shows clearly the importance of the dislocation density concept approach in elasto-plasticity.

Observe that our model is at variance with models already proposed for small elastic deformations, which essentially are defined by making the physically linear ansatz $S_2 = D.(C - C_p)$ where $D$ is a fourth order positiv definite symmetric elasticity tensor, $C = F^T F$ and $C_p$ is some plastic variable. It turns out however that the associated equilibrium equations are neither linear in $F$ nor in general elliptic. There may even be no solution of the equilibrium system due to the possible formation of microstructure. This may indicate that our approach of defining a model for small elastic deformations is more likely to lead to well posed problems and to stable numerical algorithms.

### 3. References

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