Analysis of the deformation of Cosserat elastic shells using the dislocation density tensor

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Abstract We consider the Cosserat shell approach under finite rotations. The Cosserat shell features an additional, in principle independent orthogonal frame. In this setting we establish a novel curvature tensor which we call the shell dislocation density tensor. For this variant, we derive the equations and in a hyperelastic context we show existence of minimizers under generic convexity assumptions on the elastic energies in terms of nonlinear strain measures. The correspondence between our formulation and proposals in the literature is established.

1 Introduction

Thin shell-structures still represent one of the most challenging facets of problems in nonlinear elasticity. Due to their flexibility, large rotations are a commonplace observation. The modelling definitely calls for a sound finite rotation treatment. While models based on the Kirchhoff–Love normality assumption allow for a reasonable mathematical treatment in the linear, infinitesimal strain setting, this is not the case in the finite strain setting. Here, also from an engineering point of view, models with independent director fields in the spirit of the Reissner–Mindlin kinematics are more widely used for the ease with which these shell models can be coupled to beams in structural approaches.

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The mathematics of these models with one independent director is, however, not yet settled. One of the reasons of this shortcoming is the lacking control of rotations. In the last decade, this fundamental problem has been understood [11] and as an answer, finite rotation Cosserat-shell models have been proposed and analysed. The Cosserat kinematics endows the shell with an additional orthogonal frame, in principle independent of the deformation of the shell. For this frame, new balance equations are established. For such models, under suitable convexity assumptions on strain measures which *do not* preclude buckling, existence results can be shown when formulating the model in a hyperelastic setting. Uniqueness cannot be shown and is not to be expected. With such approaches, a quite successful modelling is possible [19]. As it turns out, the basic modelling ingredients have already been known to the Cosserat brothers [6]. However, they have never discussed any constitutive assumptions. We mention that the kinematics of Cosserat shells is equivalent to the kinematics of the so-called 6-parameter shell model, see e.g. [4, 8, 16, 2].

In this contribution we provide such a discussion with a view towards a new tensor – the shell dislocation density tensor, which seems quite appropriate to express the curvature term for the orthogonal frame-field living in $\mathrm{SO}(3)$. More precisely, we present first the strain and curvature measures which are commonly used in Cosserat shell models and introduce the new dislocation density tensor, as an alternative strain measure for orientation (curvature) change. We establish the extended Nye's formula, which expresses the relationship between the shell bending-curvature tensor and the shell dislocation density tensor. Then, we write the principle of virtual work and the constitutive relations using the dislocation density tensor. We formulate the minimization problem for the deformation of Cosserat-type shells and prove an existence theorem using the direct methods of the calculus of variations. As an application of these results, we investigate the special case of isotropic elastic shells. We finish with establishing the correspondence between our new formulation and more well-known representations.

2 Strain and curvature measures in the Cosserat (6-parameter) shell model

We present shortly the kinematical model of Cosserat-type shells, which coincides with the kinematical model of 6-parameter shells, see e.g. [2, 4, 8].

2.1 Kinematics

In this model, every material point has 6 kinematical degrees of freedom: 3 for translations and 3 for rotations. To describe the rotational motion of material points we attach a triad of orthonormal vectors (called *directors*) to every point.

Let $\mathscr{S}_\xi$ be the reference configuration of a shell and consider the midsurface $\omega_\xi \subset \mathbf{R}^3$. We denote by (ξ_1, ξ_2, ξ_3) a generic point of the deformable surface ω_ξ. The Cosserat-type shell is characterized by two fields: the vectorial map $\mathbf{y}_\xi : \omega_\xi \to \omega_c$ for the deformation and the microrotation tensor $\mathbf{R}_\xi : \omega_\xi \to \mathrm{SO}(3)$. Here, ω_c represents the deformed (current) configuration of the midsurface and $\mathrm{SO}(3)$ is the group of proper orthogonal 3×3 tensors.

The reference midsurface ω_ξ admits a parametric representation

$$\mathbf{y}_0 : \omega \to \omega_\xi\,, \qquad \mathbf{y}_0(x_1,x_2) = (\xi_1, \xi_2, \xi_3),$$

where $\omega \subset \mathbf{R}^2$ is a flat domain with Lipschitz boundary $\partial\omega$. We refer the domain ω to an orthogonal Cartesian frame $Ox_1x_2x_3$ such that $\omega \subset Ox_1x_2$ and let $\mathbf{e}_i$ be the unit vectors along the coordinate axes Ox_i.

The *deformation function* $\mathbf{y}$ and the *elastic microrotation* $\mathbf{Q}_e$ are defined by the compositions

$$\begin{aligned} \mathbf{y} &= \mathbf{y}_\xi \circ \mathbf{y}_0 : \omega \to \omega_c\,, & \mathbf{y}(x_1,x_2) &:= \mathbf{y}_\xi\big(\mathbf{y}_0(x_1,x_2)\big),\\ \mathbf{Q}_e &= \mathbf{R}_\xi \circ \mathbf{y}_0 : \omega \to \mathrm{SO}(3), & \mathbf{Q}_e(x_1,x_2) &:= \mathbf{R}_\xi\big(\mathbf{y}_0(x_1,x_2)\big). \end{aligned}$$

Then, the *total microrotation* $\overline{\mathbf{R}}$ is defined by

$$\overline{\mathbf{R}} : \omega \to \mathrm{SO}(3), \qquad \overline{\mathbf{R}}(x_1,x_2) = \mathbf{Q}_e(x_1,x_2)\,\mathbf{Q}_0(x_1,x_2),$$

where $\mathbf{Q}_0 : \omega \to \mathrm{SO}(3)$ is the *initial microrotation*, which describes the orientation of points in the reference configuration. With the help of the directors, one can express the microrotation tensors in the following way

$$\mathbf{Q}_e = \mathbf{d}_i \otimes \mathbf{d}_i^0\,, \qquad \overline{\mathbf{R}} = \mathbf{Q}_e\,\mathbf{Q}_0 = \mathbf{d}_i \otimes \mathbf{e}_i\,, \qquad \mathbf{Q}_0 = \mathbf{d}_i^0 \otimes \mathbf{e}_i\,, \tag{1}$$

where $\mathbf{d}_i^0$ stand for the initial directors (attached to points in ω_ξ) and $\mathbf{d}_i$ are the directors in the deformed configuration ω_c $(i = 1,2,3)$. In relation (1) and throughout the paper we employ the usual conventions for indices: the Latin indices $i, j, k, \ldots$ range over the set $\{1,2,3\}$, while the Greek indices $\alpha, \beta, \gamma, \ldots$ are confined to the range $\{1,2\}$; the comma preceding an index i denotes partial derivatives with respect to x_i; the Einstein summation convention over repeated indices is also used.

2.2 Differential geometry

For the differential geometry of the reference surface ω_ξ we introduce some notations. Let $\mathbf{a}_\alpha$ be the covariant base vectors and $\mathbf{a}^\beta$ be the contravariant base vectors in the tangent plane: $\mathbf{a}_\alpha := \mathbf{y}_{0,\alpha}$ and $\mathbf{a}_\alpha \cdot \mathbf{a}^\beta = \delta_\alpha^\beta$ (the Kronecker delta). We denote

$$a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta\,, \qquad a^{\alpha\beta} := \mathbf{a}^\alpha \cdot \mathbf{a}^\beta, \qquad a = \sqrt{\det\big(a_{\alpha\beta}\big)_{2\times 2}} = |\mathbf{a}_1 \times \mathbf{a}_2| > 0.$$

The unit normal $\mathbf{n}_0$ to the surface is given by

$$\mathbf{n}_0 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{a}$$

and we also use $\mathbf{a}_3 := \mathbf{a}^3 := \mathbf{n}_0$. The surface gradient Grad_s and surface divergence Div_s operators are defined for a vector field $\mathbf{v}$ by

$$\mathrm{Grad}_s\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_\alpha} \otimes \mathbf{a}^\alpha = \mathbf{v}_{,\alpha} \otimes \mathbf{a}^\alpha, \qquad \mathrm{Div}_s\,\mathbf{v} = \mathrm{tr}\big[\mathrm{Grad}_s\,\mathbf{v}\big] = \mathbf{v}_{,\alpha} \cdot \mathbf{a}^\alpha. \tag{2}$$

In [3] we have introduced the surface Curl operator curl_s for vector fields $\mathbf{v}$ and, respectively, Curl_s for tensor fields $\mathbf{T}$ by

$$\begin{aligned} \big(\mathrm{curl}_s\,\mathbf{v}\big) \cdot \mathbf{k} &:= \mathrm{Div}_s\big(\mathbf{v} \times \mathbf{k}\big) \qquad \text{for all constant vectors } \mathbf{k}, \\ \big(\mathrm{Curl}_s\,\mathbf{T}\big)^T \mathbf{k} &:= \mathrm{curl}_s\big(\mathbf{T}^T \mathbf{k}\big) \qquad \text{for all constant vectors } \mathbf{k}. \end{aligned} \tag{3}$$

In view of these definitions, we have the relations

$$\mathrm{curl}_s\,\mathbf{v} = -\mathbf{v}_{,\alpha} \times \mathbf{a}^\alpha, \qquad \mathrm{Curl}_s\,\mathbf{T} = -\mathbf{T}_{,\alpha} \times \mathbf{a}^\alpha. \tag{4}$$

Using these notations we can express the *first fundamental tensor* $\mathbf{a}$ and the *second fundamental tensor* $\mathbf{b}$ of the surface ω_ξ in the forms

$$\begin{aligned} \mathbf{a} &= a_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta = a^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha, \\ \mathbf{b} &= -\mathrm{Grad}_s\,\mathbf{n}_0 = -\mathbf{n}_{0,\alpha} \otimes \mathbf{a}^\alpha = b_{\alpha\beta}\,\mathbf{a}^\alpha \otimes \mathbf{a}^\beta = b^\alpha_\beta\,\mathbf{a}_\alpha \otimes \mathbf{a}^\beta, \\ \text{with} &\qquad b_{\alpha\beta} = -\mathbf{n}_{0,\beta} \cdot \mathbf{a}_\alpha = b_{\beta\alpha}\,, \qquad b^\alpha_\beta = -\mathbf{n}_{0,\beta} \cdot \mathbf{a}^\alpha\,. \end{aligned}$$

Remark 1. The initial directors $\mathbf{d}_i^0$ are usually chosen such that

$$\mathbf{d}_3^0 = \mathbf{n}_0\,, \qquad \mathbf{d}_\alpha^0 \cdot \mathbf{n}_0 = 0\,, \tag{5}$$

i.e. $\mathbf{d}_3^0$ is orthogonal to ω_ξ and $\mathbf{d}_\alpha^0$ belong to the tangent plane. This assumption is not necessary in general, but it will be adopted here since it simplifies many of the subsequent expressions. In the deformed configuration, the director $\mathbf{d}_3$ is no longer orthogonal to the surface ω_c (the Kirchhof-Love condition is not imposed). One convenient choice of the initial microrotation tensor $\mathbf{Q}_0 = \mathbf{d}_i \otimes \mathbf{e}_i$ such that the conditions (5) be satisfied is (see Remark 10 of [1])

$$\mathbf{Q}_0 = \mathrm{polar}\big(\mathbf{a}_i \otimes \mathbf{e}_i\big),$$

where polar($\mathbf{T}$) denotes the orthogonal tensor given by the polar decomposition of any tensor $\mathbf{T}$.

2.3 The dislocation density tensor

In view of (2), the surface gradient of the deformation is

$$\mathbf{F} := \mathrm{Grad}_s\, \mathbf{y} \;=\; \mathbf{y}_{,\alpha} \otimes \mathbf{a}^{\alpha}.$$

The *elastic shell strain tensor* $\mathbf{E}_e$ is defined by [4, 8]

$$\mathbf{E}_e \;=\; \mathbf{Q}_e^T\, \mathrm{Grad}_s \mathbf{y} - \mathbf{a} = \big(\mathbf{Q}_e^T\, \mathbf{y}_{,\alpha} - \mathbf{y}_{0,\alpha}\big) \otimes \mathbf{a}^{\alpha} \tag{6}$$

and the *elastic shell bending-curvature tensor* $\mathbf{K}_e$ is [2, 4, 8]

$$\mathbf{K}_e = \mathrm{axl}\big(\mathbf{Q}_e^T \mathbf{Q}_{e,\alpha}\big) \otimes \mathbf{a}^{\alpha} = \mathbf{Q}_0\big[\mathrm{axl}\big(\overline{\mathbf{R}}^T \overline{\mathbf{R}}_{,\alpha}\big) - \mathrm{axl}\big(\mathbf{Q}_0^T \mathbf{Q}_{0,\alpha}\big)\big], \tag{7}$$

which is a measure of orientation (curvature) change for Cosserat shells.

In [3] we have defined the *dislocation density tensor* $\mathbf{D}_e$ by

$$\mathbf{D}_e = \mathbf{Q}_e^T\, \mathrm{Curl}_s\, \mathbf{Q}_e\; . \tag{8}$$

Using the formula $(4)_2$, we can write this definition in the form

$$\mathbf{D}_e = \mathbf{Q}_e^T \big(-\mathbf{Q}_{e,\alpha} \times \mathbf{a}^{\alpha}\big) = -\big(\mathbf{Q}_e^T \mathbf{Q}_{e,\alpha}\big) \times \mathbf{a}^{\alpha}. \tag{9}$$

We have shown that the tensor $\mathbf{D}_e$ represents an alternative strain measure for orientation (curvature) change. Indeed, in [3] we have established the following relation

$$\mathbf{D}_e = -\mathbf{K}_e^T + \big(\mathrm{tr}\,\mathbf{K}_e\big)\mathbf{1}_3 \qquad \text{or equivalently,} \qquad \mathbf{K}_e = -\mathbf{D}_e^T + \frac{1}{2}\big(\mathrm{tr}\,\mathbf{D}_e\big)\mathbf{1}_3\,, \tag{10}$$

where $\mathbf{1}_3$ is the identity tensor in the Euclidean 3-space. We call (10) the *extended Nye's formula*, since it is a generalization of a well-known formula for infinitesimal strains in three-dimensional elasticity [14]. The formula (10) expresses the relationship between the shell bending-curvature tensor $\mathbf{K}_e$ and the dislocation density tensor $\mathbf{D}_e$.

We present next a new proof of the extended Nye's formula (10). This proof is simpler as the one shown in [3] and is based on the relation

$$\mathbf{A} \times \mathbf{v} = \mathbf{v} \otimes \mathrm{axl}\,\mathbf{A} - \big(\mathbf{v} \cdot \mathrm{axl}\,\mathbf{A}\big)\mathbf{1}_3\,, \tag{11}$$

which holds true for every vector $\mathbf{v}$ and any skew-symmetric second order tensor $\mathbf{A}$.

Proof of relation (11) : It is well-known that $\mathbf{A} = \mathbf{1}_3 \times \mathrm{axl}\,\mathbf{A}$, where $\mathrm{axl}\,\mathbf{A}$ is the axial vector of the skew-symmetric tensor $\mathbf{A}$. Then, we can write

$$\begin{aligned}
\mathbf{A} \times \mathbf{v} \;&=\; \big(\mathbf{1}_3 \times \mathrm{axl}\,\mathbf{A}\big) \times \mathbf{v} = \big[(\mathbf{e}_i \otimes \mathbf{e}_i) \times \mathrm{axl}\,\mathbf{A}\big] \times \mathbf{v} \\
&= \mathbf{e}_i \otimes \big[(\mathbf{e}_i \times \mathrm{axl}\,\mathbf{A}) \times \mathbf{v}\big] = \mathbf{e}_i \otimes \big[(\mathbf{e}_i \cdot \mathbf{v})\,\mathrm{axl}\,\mathbf{A} - (\mathbf{v} \cdot \mathrm{axl}\,\mathbf{A})\,\mathbf{e}_i\big] \\
&= \big[(\mathbf{v} \cdot \mathbf{e}_i)\,\mathbf{e}_i\big] \otimes \mathrm{axl}\,\mathbf{A} - (\mathbf{v} \cdot \mathrm{axl}\,\mathbf{A})(\mathbf{e}_i \otimes \mathbf{e}_i) = \mathbf{v} \otimes \mathrm{axl}\,\mathbf{A} - \big(\mathbf{v} \cdot \mathrm{axl}\,\mathbf{A}\big)\mathbf{1}_3\,.
\end{aligned}$$

Proof of the extended Nye's formula (10) : We apply the relation (11) for the skew-symmetric tensor $\mathbf{A}=-\mathbf{Q}_e^T\mathbf{Q}_{e,\alpha}$ and the vector $\mathbf{v}=\mathbf{a}^\alpha$.
Upon summation over $\alpha=1,2$ we obtain from (7), (9) and (11) that

$$\begin{aligned}\mathbf{D}_e &= \big(-\mathbf{Q}_e^T\mathbf{Q}_{e,\alpha}\big)\times\mathbf{a}^\alpha = -\mathbf{a}^\alpha\otimes\operatorname{axl}\big(\mathbf{Q}_e^T\mathbf{Q}_{e,\alpha}\big)+\big(\mathbf{a}^\alpha\cdot\operatorname{axl}(\mathbf{Q}_e^T\mathbf{Q}_{e,\alpha})\big)\mathbf{1}_3\\ &= -\mathbf{K}_e^T+\big(\operatorname{tr}\mathbf{K}_e\big)\mathbf{1}_3\,,\end{aligned}$$

which concludes the proof.

Remark 2. We can write the extended Nye's formula (10) in an equivalent form, if we make use of the following orthogonal decomposition:
Any second order tensor $\mathbf{X}\in\mathbb{R}^{3\times3}$ can be decomposed as direct sum in the form (the Cartan–Lie–algebra decomposition)

$$\mathbf{X} = \operatorname{dev}_3\operatorname{sym}\mathbf{X}+\operatorname{skew}\mathbf{X}+\frac{1}{3}\big(\operatorname{tr}\mathbf{X}\big)\mathbf{1}_3\,, \tag{12}$$

where $\operatorname{sym}\mathbf{S}$ is the symmetric part, $\operatorname{skew}\mathbf{S}$ the skew-symmetric part and $\operatorname{dev}_3\mathbf{S}:=\mathbf{S}-\frac{1}{3}(\operatorname{tr}\mathbf{S})\mathbf{1}_3$ is the deviatoric part of any second order tensor $\mathbf{S}$.

If we apply the operators tr, skew and $\operatorname{dev}_3\operatorname{sym}$ to the relation (10) we obtain, respectively

$$\operatorname{tr}\mathbf{D}_e=2\operatorname{tr}\mathbf{K}_e\,,\quad \operatorname{skew}\mathbf{D}_e=\operatorname{skew}\mathbf{K}_e\,,\quad \operatorname{dev}_3\operatorname{sym}\mathbf{D}_e=-\operatorname{dev}_3\operatorname{sym}\mathbf{K}_e\,, \tag{13}$$

which express anew the relationship between $\mathbf{K}_e$ and $\mathbf{D}_e$ and are equivalent to the extended Nye's formula (10). As a direct consequence of (13) we deduce

$$\|\mathbf{D}_e\|^2=\|\mathbf{K}_e\|^2+\big(\operatorname{tr}\mathbf{K}_e\big)^2 \qquad\text{and}\qquad \|\mathbf{K}_e\|\le\|\mathbf{D}_e\|\le 2\,\|\mathbf{K}_e\|. \qquad \square \tag{14}$$

If we analyze the structure of the dislocation density tensor $\mathbf{D}_e$ we find that

$$\mathbf{D}_e = \mathbf{a}\,\mathbf{D}_e+\operatorname{tr}\big(\mathbf{a}\,\mathbf{D}_e\big)\,\mathbf{n}_0\otimes\mathbf{n}_0\,. \tag{15}$$

Thus, we see that the essential part of the tensor $\mathbf{D}_e$ is the tensor $\mathbf{a}\,\mathbf{D}_e=D_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta+D_{\alpha3}\,\mathbf{a}^\alpha\otimes\mathbf{n}_0$ (which has only 6 non-vanishing components $D_{\alpha i}$). Indeed, in view of (15) the two components of $\mathbf{D}_e$ in the directions $\mathbf{n}_0\otimes\mathbf{a}^\alpha$ are zero, while the component in the direction $\mathbf{n}_0\otimes\mathbf{n}_0$ is equal to $\operatorname{tr}\big(\mathbf{a}\,\mathbf{D}_e\big)$. In other words, all the information carried by the dislocation density tensor $\mathbf{D}_e$ is already contained in its part $\mathbf{a}\,\mathbf{D}_e$. For this reason, we define the new *shell dislocation density tensor* $\mathbf{D}_s$ by

$$\mathbf{D}_s:=\mathbf{a}\,\mathbf{D}_e=\mathbf{a}\,\mathbf{Q}_e^T\operatorname{Curl}_s\mathbf{Q}_e=-\mathbf{a}\big(\mathbf{Q}_e^T\mathbf{Q}_{e,\alpha}\big)\times\mathbf{a}^\alpha, \tag{16}$$

which is more appropriate for the shell theory, since it is a tensor in the space $T_x\otimes E$ (where T_x is the tangent plane and E is the Euclidean 3-space). The definition (16) can be formulated as follows: by multiplication from the left with $\mathbf{a}$ we take the projection of the left leg of $\mathbf{D}_e$ on the tangent plane and obtain the shell dislocation density tensor $\mathbf{D}_s$.

Next, we would like to present the relationship between the shell dislocation density tensor $\mathbf{D}_s$ and the shell bending-curvature tensor $\mathbf{K}_e$: if we multiply (10) with $\mathbf{a}$ and use the relations $\mathbf{a}\,\mathbf{D}_e=\mathbf{D}_s$ and $\mathbf{K}_e\,\mathbf{a}=\mathbf{K}_e$ we obtain

$$\mathbf{D}_s=-\mathbf{K}_e^T+\big(\mathrm{tr}\,\mathbf{K}_e\big)\mathbf{a} \qquad \text{or equivalently,} \qquad \mathbf{K}_e=-\mathbf{D}_s^T+\big(\mathrm{tr}\,\mathbf{D}_s\big)\mathbf{a}. \tag{17}$$

The relations (17) express the *extended Nye's formula* for the tensors $\mathbf{D}_s$ and $\mathbf{K}_e$. We observe that, in contrast to (10), the two relations (17) have a reciprocal structure: the two tensors $\mathbf{D}_s$ and $\mathbf{K}_e$ play symmetrical roles in (17).

Furthermore, we can establish an equivalent representation of the extended Nye's formula (17) using a corresponding orthogonal decomposition for these tensors (as a counterpart of equations (12) and (13)):

Lemma 1. *Any second order tensor* $\mathbf{X}$ *which satisfies the condition* $\langle\mathbf{X},\mathbf{n}_0\otimes\mathbf{n}_0\rangle=0$ *(i.e. the component of* $\mathbf{X}$ *in the direction* $\mathbf{n}_0\otimes\mathbf{n}_0$ *is zero) can be decomposed as a direct sum in the form*

$$\mathbf{X}=\mathrm{dev_s sym}\,\mathbf{X}+\mathrm{skew}\,\mathbf{X}+\frac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\mathbf{a}\,, \tag{18}$$

where we denote by $\mathrm{dev_s}\,\mathbf{S}:=\mathbf{S}-\frac{1}{2}\big(\mathrm{tr}\,\mathbf{S}\big)\,\mathbf{a}$ *the* surface deviatoric part *of any second order tensor* $\mathbf{S}$.

Proof. In view of the definition of the oprator $\mathrm{dev_s}$, we have

$$\begin{aligned}\mathrm{dev_s sym}\,\mathbf{X}+\mathrm{skew}\,\mathbf{X}+\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\mathbf{a}&=\mathrm{sym}\,\mathbf{X}-\tfrac{1}{2}\,\mathrm{tr}\big(\mathrm{sym}\,\mathbf{X}\big)\mathbf{a}\\&\quad+\mathrm{skew}\,\mathbf{X}+\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\mathbf{a}=\mathrm{sym}\,\mathbf{X}+\mathrm{skew}\,\mathbf{X}=\mathbf{X}\,.\end{aligned}$$

We see immediately that

$$\langle\,\mathrm{dev_s sym}\,\mathbf{X}\,,\,\mathrm{skew}\,\mathbf{X}\,\rangle=0 \qquad \text{and} \qquad \langle\,\mathrm{skew}\,\mathbf{X}\,,\,\frac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\mathbf{a}\,\rangle=0,$$

since $\mathrm{dev_s sym}\,\mathbf{X}$ and $\frac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\mathbf{a}$ are symmetric and $\mathrm{skew}\,\mathbf{X}$ is skew-symmetric. Finally, we employ the relations $\mathbf{a}=\mathbf{1}_3-\mathbf{n}_0\otimes\mathbf{n}_0$ and $\langle\mathbf{X},\mathbf{n}_0\otimes\mathbf{n}_0\rangle=0$ to get

$$\begin{aligned}\langle\,\mathrm{dev_s sym}\,\mathbf{X}\,,\,\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\mathbf{a}\,\rangle&=\langle\,\mathrm{sym}\,\mathbf{X}-\tfrac{1}{2}\,\mathrm{tr}\big(\mathrm{sym}\,\mathbf{X}\big)\mathbf{a}\,,\,\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\mathbf{a}\,\rangle\\&=\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\langle\,\mathbf{X}-\tfrac{1}{2}\big(\mathrm{tr}\mathbf{X}\big)\mathbf{a}\,,\,\mathbf{a}\,\rangle=\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\langle\,\mathbf{X}\,,\,\mathbf{a}\,\rangle-\tfrac{1}{4}\big(\mathrm{tr}\,\mathbf{X}\big)^2\,\langle\,\mathbf{a}\,,\,\mathbf{a}\,\rangle\\&=\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\langle\,\mathbf{X}\,,\,\mathbf{1}_3-\mathbf{n}_0\otimes\mathbf{n}_0\,\rangle-\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)^2\\&=\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)^2-\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)\,\langle\,\mathbf{X}\,,\,\mathbf{n}_0\otimes\mathbf{n}_0\,\rangle-\tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{X}\big)^2=0.\end{aligned}$$ □

Remark 3. In view of the definitions (7) and (16) we see that the tensors $\mathbf{K}_e$ and $\mathbf{D}_s$ satisfy the conditions $\langle\mathbf{K}_e,\mathbf{n}_0\otimes\mathbf{n}_0\rangle=0$ and, respectively, $\langle\mathbf{D}_s,\mathbf{n}_0\otimes\mathbf{n}_0\rangle=0$. Applying the above Lemma we obtain the orthogonal decompositions

$$
\begin{aligned}
\mathbf{K}_e &= \mathrm{dev}_s\mathrm{sym}\,\mathbf{K}_e + \mathrm{skew}\,\mathbf{K}_e + \tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{K}_e\big)\,\mathbf{a}\,,\\
\mathbf{D}_s &= \mathrm{dev}_s\mathrm{sym}\,\mathbf{D}_s + \mathrm{skew}\,\mathbf{D}_s + \tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{D}_s\big)\,\mathbf{a}\,.
\end{aligned}
\tag{19}
$$

If we insert the relations (19) into (17) and compare the corresponding terms, then we obtain (in view of the uniqueness of the representation given by Lemma 1) the following equivalent form of the extended Nye's formula (17):

$$
\mathrm{tr}\,\mathbf{D}_s = \mathrm{tr}\,\mathbf{K}_e\,, \quad \mathrm{skew}\,\mathbf{D}_s = \mathrm{skew}\,\mathbf{K}_e\,, \quad \mathrm{dev}_s\mathrm{sym}\,\mathbf{D}_s = -\mathrm{dev}_s\mathrm{sym}\,\mathbf{K}_e\,. \tag{20}
$$

As a direct consequence of these equations we find

$$
\|\mathbf{D}_s\| = \|\mathbf{K}_e\|\,. \tag{21}
$$

Indeed, by virtue of the orthogonal decompositions (19) and the relations (20) we get

$$
\begin{aligned}
\|\mathbf{D}_s\|^2 &= \|\mathrm{dev}_s\mathrm{sym}\,\mathbf{D}_s\|^2 + \|\mathrm{skew}\,\mathbf{D}_s\|^2 + \tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{D}_s\big)^2\\
&= \|\mathrm{dev}_s\mathrm{sym}\,\mathbf{K}_e\|^2 + \|\mathrm{skew}\,\mathbf{K}_e\|^2 + \tfrac{1}{2}\big(\mathrm{tr}\,\mathbf{K}_e\big)^2 = \|\mathbf{K}_e\|^2.
\end{aligned}
$$

3 Governing equations for the equilibrium of Cosserat shells

In what follows, we present the field equations which govern the deformation of Cosserat shells. We express the principle of virtual work using the shell dislocation density tensor $\mathbf{D}_s$. Then, we introduce the stress measure which is work-conjugate to $\mathbf{D}_s$ and deduce the corresponding constitutive equations.

Let $\mathbf{N}$ be the internal surface stress tensor and $\mathbf{M}$ be the internal surface couple tensor (of the first Piola-Kirchhoff type) for the shell. Then, the local equilibrium equations can be expressed in the form (see e.g., [8, 1])

$$
\mathrm{Div}_s\,\mathbf{N} + \mathbf{f} = \mathbf{0}, \qquad \mathrm{Div}_s\,\mathbf{M} + \mathrm{axl}(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T) + \mathbf{c} = \mathbf{0}, \tag{22}
$$

where $\mathbf{f}$ and $\mathbf{c}$ are the external surface force and couple vectors.

Let ν be the external unit normal vector to the boundary curve $\partial\omega_\xi$ lying in the tangent plane. We assume boundary conditions of the type [7, 16]

$$
\begin{aligned}
\mathbf{N}\nu &= \mathbf{n}^*, & \mathbf{M}\nu &= \mathbf{m}^* \quad \text{along } \partial\omega_f\,,\\
\mathbf{y} &= \mathbf{y}^*, & \overline{\mathbf{R}} &= \mathbf{R}^* \quad \text{along } \partial\omega_d\,,
\end{aligned}
\tag{23}
$$

where $\partial\omega_f$ and $\partial\omega_d$ build a disjoint partition of $\partial\omega_\xi$ with length$(\partial\omega_d) > 0$. Here, $\mathbf{n}^*$ and $\mathbf{m}^*$ are the external boundary resultant force and couple vectors respectively, applied along the deformed boundary $\partial\omega_c$, but measured per unit length of $\partial\omega_\xi$.

To obtain the principle of virtual work, we consider two arbitrary smooth vector fields $\mathbf{v}$ and $\mathbf{w}$ given on ω_ξ. By multiplying the equations (22), (23) with $\mathbf{v}$ and $\mathbf{w}$, we can set the integral identity

$$\int_{\omega_\xi} \big\{ \big(\mathrm{Div}_s\,\mathbf{N}+\mathbf{f}\big)\cdot\mathbf{v}+\big[\mathrm{Div}_s\,\mathbf{M}+\mathrm{axl}(\mathbf{N}\mathbf{F}^T-\mathbf{F}\mathbf{N}^T)+\mathbf{c}\big]\cdot\mathbf{w}\big\}\,\mathrm{d}a$$
$$=\int_{\partial\omega_f}\big[\big(\mathbf{N}\nu-\mathbf{n}^*\big)\cdot\mathbf{v}+\big(\mathbf{M}\nu-\mathbf{m}^*\big)\cdot\mathbf{w}\big]\,\mathrm{d}\ell,$$

where $\mathrm{d}a$ is the area element on the surface ω_ξ and $\mathrm{d}\ell$ is the length element along $\partial\omega_f$. After some transformations we can rewrite this identity in the following form (see e.g., [8])

$$\int_{\omega_\xi}\Big(\big\langle\mathbf{N},\mathrm{Grad}_s\,\mathbf{v}-\mathbf{W}\,\mathbf{F}\big\rangle+\big\langle\mathbf{M},\mathrm{Grad}_s\,\mathbf{w}\big\rangle\Big)\mathrm{d}a=\int_{\omega_\xi}\big(\mathbf{f}\cdot\mathbf{v}+\mathbf{c}\cdot\mathbf{w}\big)\mathrm{d}a$$
$$+\int_{\partial\omega_f}\big(\mathbf{n}^*\cdot\mathbf{v}+\mathbf{m}^*\cdot\mathbf{w}\big)\mathrm{d}\ell+\int_{\partial\omega_d}\big[\big(\mathbf{N}\nu\big)\cdot\mathbf{v}+\big(\mathbf{M}\nu\big)\cdot\mathbf{w}\big]\mathrm{d}\ell, \tag{24}$$

where $\mathbf{W}=\mathbf{w}\times\mathbf{1}_3$ is the skew-symmetric tensor corresponding to the axial vector $\mathbf{w}$. Now, if we interpret $\mathbf{v}$ as the kinematically admissible virtual translation and $\mathbf{w}$ as the kinematically admissible virtual rotation of the shell, i.e.

$$\mathbf{v}=\delta\,\mathbf{y}\qquad\text{and}\qquad\mathbf{w}=\mathrm{axl}\big((\delta\mathbf{Q}_e)\mathbf{Q}_e^T\big), \tag{25}$$

then from the boundary conditions $(23)_{3,4}$ we obtain

$$\mathbf{v}=\mathbf{0},\qquad\mathbf{w}=\mathbf{0}\quad\text{along }\partial\omega_d\,,$$

which shows that the last integral in (24) vanishes. The remaining two integrals in the right-hand side of (24) describe the external virtual work and the integral in the left-hand side represents the internal virtual work. Thus, the internal virtual work power (density) $\mathscr{P}$ is given by

$$\mathscr{P}\,=\,\big\langle\mathbf{N},\mathrm{Grad}_s\,\mathbf{v}-\mathbf{W}\,\mathbf{F}\big\rangle+\big\langle\mathbf{M},\mathrm{Grad}_s\,\mathbf{w}\big\rangle. \tag{26}$$

This expression can be written in terms of the shell strain measures $\mathbf{E}_e$ and $\mathbf{K}_e$. Using transformations similar to those presented in [17, Sect.4], we can put (26) in the form

$$\mathscr{P}\,=\,\big\langle\mathbf{Q}_e^T\mathbf{N},\,\delta\,\mathbf{E}_e\big\rangle+\big\langle\mathbf{Q}_e^T\mathbf{M},\,\delta\,\mathbf{K}_e\big\rangle. \tag{27}$$

Here, $\mathbf{Q}_e^T\mathbf{N}$ and $\mathbf{Q}_e^T\mathbf{M}$ are the shell stress measures in the material representation. They are work-conjugate to the shell strain measures $\mathbf{E}_e$ and $\mathbf{K}_e$, respectively. By virtue of (27), the principle of virtual work (24) becomes

$$\int_{\omega_\xi}\Big(\big\langle\mathbf{Q}_e^T\mathbf{N},\,\delta\,\mathbf{E}_e\big\rangle+\big\langle\mathbf{Q}_e^T\mathbf{M},\,\delta\,\mathbf{K}_e\big\rangle\Big)\,\mathrm{d}a$$
$$=\int_{\omega_\xi}\big(\mathbf{f}\cdot\mathbf{v}+\mathbf{c}\cdot\mathbf{w}\big)\mathrm{d}a+\int_{\partial\omega_f}\big(\mathbf{n}^*\cdot\mathbf{v}+\mathbf{m}^*\cdot\mathbf{w}\big)\mathrm{d}\ell, \tag{28}$$

where $\mathbf{v}$ and $\mathbf{w}$ are given by (25).

The corresponding constitutive equations (under hyperelasticity assumptions) are

$$\mathbf{Q}_e^T\mathbf{N} = \frac{\partial W}{\partial \mathbf{E}_e} \qquad \text{and} \qquad \mathbf{Q}_e^T\mathbf{M} = \frac{\partial W}{\partial \mathbf{K}_e}\,, \tag{29}$$

where $W = W(\mathbf{E}_e, \mathbf{K}_e)$ is the elastically stored energy density for Cosserat shells.

We show next that the internal virtual work power (26) can also be expressed using the shell dislocation density tensor $\mathbf{D}_s$. On the basis of the extended Nye's formula $(17)_2$ and the relation $\big\langle \mathbf{Q}_e^T\mathbf{M}, \mathbf{a} \big\rangle = \mathrm{tr}\big(\mathbf{Q}_e^T\mathbf{M}\big)$, we get

$$\begin{aligned}
\big\langle \mathbf{Q}_e^T\mathbf{M}, \delta\mathbf{K}_e \big\rangle &= \big\langle \mathbf{Q}_e^T\mathbf{M}, -\delta\mathbf{D}_s^T + \big(\mathrm{tr}\,\delta\mathbf{D}_s\big)\,\mathbf{a} \big\rangle \\
&= -\big\langle \mathbf{Q}_e^T\mathbf{M}, \delta\mathbf{D}_s^T \big\rangle + \mathrm{tr}\big(\mathbf{Q}_e^T\mathbf{M}\big)\,\mathrm{tr}\big(\delta\mathbf{D}_s\big) \\
&= \big\langle -\big(\mathbf{Q}_e^T\mathbf{M}\big)^T + \mathrm{tr}\big(\mathbf{Q}_e^T\mathbf{M}\big)\,\mathbf{a}\,,\, \delta\mathbf{D}_s \big\rangle.
\end{aligned}$$

Inserting this expression into (27) we find

$$\mathscr{P} = \big\langle \mathbf{Q}_e^T\mathbf{N}, \delta\mathbf{E}_e \big\rangle + \big\langle -\big(\mathbf{Q}_e^T\mathbf{M}\big)^T + \mathrm{tr}\big(\mathbf{Q}_e^T\mathbf{M}\big)\,\mathbf{a}\,,\, \delta\mathbf{D}_s \big\rangle$$

and hence, the work-conjugate shell stress measures to $\mathbf{D}_s$ is $-\big(\mathbf{Q}_e^T\mathbf{M}\big)^T + \mathrm{tr}\big(\mathbf{Q}_e^T\mathbf{M}\big)\,\mathbf{a}$.

If we write the elastically stored energy density (strain energy density) W as a function $W = \widehat{W}(\mathbf{E}_e, \mathbf{D}_s)$, then the corresponding constitutive equations are

$$\mathbf{Q}_e^T\mathbf{N} = \frac{\partial \widehat{W}}{\partial \mathbf{E}_e} \qquad \text{and} \qquad -\big(\mathbf{Q}_e^T\mathbf{M}\big)^T + \mathrm{tr}\big(\mathbf{Q}_e^T\mathbf{M}\big)\,\mathbf{a} = \frac{\partial \widehat{W}}{\partial \mathbf{D}_s}\,. \tag{30}$$

The last relation can be inverted (in the same way as the extended Nye's formula (17)) and (30) is equivalent to

$$\mathbf{N} = \mathbf{Q}_e\,\frac{\partial \widehat{W}}{\partial \mathbf{E}_e} \qquad \text{and} \qquad \mathbf{M} = \mathbf{Q}_e\Big[-\Big(\frac{\partial \widehat{W}}{\partial \mathbf{D}_s}\Big)^T + \Big(\mathrm{tr}\,\frac{\partial \widehat{W}}{\partial \mathbf{D}_s}\Big)\,\mathbf{a}\Big]. \tag{31}$$

4 Variational formulation and existence of minimizers

We consider the usual Lebesgue spaces $L^p(\omega, \mathbb{R}^3)$, $L^p(\omega, \mathbb{R}^{3\times 3})$ with $p \geq 1$ and the Sobolev spaces $H^1(\omega, \mathbb{R}^3)$, $H^1(\omega, \mathbb{R}^{3\times 3})$. With abuse of notation, we introduce the subset

$$L^p\big(\omega, \mathrm{SO}(3)\big) := \big\{\mathbf{Q} \in L^p(\omega, \mathbb{R}^{3\times 3}) \,|\, \mathbf{Q}(x_1,x_2) \in \mathrm{SO}(3) \ \text{ for a.e. } (x_1,x_2) \in \omega\big\}$$

with the induced strong topology of $L^p(\omega, \mathbb{R}^{3\times 3})$ and the subset

$$H^1\big(\omega, \mathrm{SO}(3)\big) := \big\{\mathbf{Q} \in H^1(\omega, \mathbb{R}^{3\times 3}) \,|\, \mathbf{Q}(x_1,x_2) \in \mathrm{SO}(3) \ \text{ for a.e. } (x_1,x_2) \in \omega\big\}$$

with the induced strong and weak topologies of $H^1(\omega, \mathbb{R}^{3\times 3})$.

Let us define the *admissible set* $\mathscr{A}$ by

$$\mathscr{A} = \big\{ (\mathbf{y}, \overline{\mathbf{R}}) \in H^1(\omega, \mathbb{R}^3) \times H^1\big(\omega, \mathrm{SO}(3)\big) \;\big|\; \mathbf{y}_{\big|\partial\omega_d} = \mathbf{y}^*,\; \overline{\mathbf{R}}_{\big|\partial\omega_d} = \mathbf{R}^* \big\}, \tag{32}$$

where the boundary data satisfy $\mathbf{y}^* \in H^1(\omega, \mathbb{R}^3)$ and $\mathbf{R}^* \in H^1\big(\omega, \mathrm{SO}(3)\big)$.

We consider the *total energy functional*

$$\begin{aligned} \mathscr{E}(\mathbf{y}, \overline{\mathbf{R}}) &= \int_{\omega_\xi} \widehat{W}(\mathbf{E}_e, \mathbf{D}_s)\, \mathrm{d}a - \Lambda(\mathbf{y}, \overline{\mathbf{R}}) \\ &= \int_{\omega} \widehat{W}(\mathbf{E}_e, \mathbf{D}_s)\, a(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 - \Lambda(\mathbf{y}, \overline{\mathbf{R}})\,, \end{aligned} \tag{33}$$

for every $(\mathbf{y}, \overline{\mathbf{R}}) \in \mathscr{A}$. Here, $\Lambda(\mathbf{y}, \overline{\mathbf{R}})$ denotes the *external loading potential*

$$\Lambda(\mathbf{y}, \overline{\mathbf{R}}) = \int_{\omega_\xi} \mathbf{f} \cdot \mathbf{u}\, \mathrm{d}a + \Pi_{\omega_\xi}(\overline{\mathbf{R}}) + \int_{\partial\omega_f} \mathbf{n}^* \cdot \mathbf{u}\, \mathrm{d}\ell + \Pi_{\partial\omega_f}(\overline{\mathbf{R}}), \tag{34}$$

where $\mathbf{u} := \mathbf{y} - \mathbf{y}_0$ is the displacement vector and we assume that $\mathbf{f} \in L^2(\omega, \mathbb{R}^3)$ and $\mathbf{n}^* \in L^2(\partial\omega_f, \mathbb{R}^3)$. The potential $\Pi_{\omega_\xi} : L^2\big(\omega, \mathrm{SO}(3)\big) \to \mathbb{R}$ of the external surface couples $\mathbf{c}$ and the potential $\Pi_{\partial\omega_f} : L^2\big(\partial\omega_f, \mathrm{SO}(3)\big) \to \mathbb{R}$ of the external boundary couples $\mathbf{m}^*$ are assumed to be continuous and bounded operators.

We formulate the following two-field minimization problem: *find the pair* $(\hat{\mathbf{y}}, \hat{\mathbf{R}}) \in \mathscr{A}$ *which realizes the minimum of the total energy functional* $\mathscr{E}(\mathbf{y}, \overline{\mathbf{R}})$ *given by (33).*

We can prove the following existence result

Theorem 1. *Assume that the reference configuration of the Cosserat shell satisfies the regularity conditions*

$$\begin{aligned} \mathbf{y}_0 \in H^1(\omega, \mathbb{R}^3), \qquad & \mathbf{Q}_0 \in H^1(\omega, \mathrm{SO}(3)), \\ \mathbf{a}_\alpha = \mathbf{y}_{0,\alpha} \in L^\infty(\omega, \mathbb{R}^3), \qquad & a(x_1, x_2) \geq a_0 > 0, \end{aligned} \tag{35}$$

where a_0 is a constant. Moreover, the strain energy density $\widehat{W}(\mathbf{E}_e, \mathbf{D}_s)$ is assumed to be a quadratic convex function of $(\mathbf{E}_e, \mathbf{D}_s)$, which is also coercive, i.e. there exists a constant $C_0 > 0$ such that

$$\widehat{W}(\mathbf{E}_e, \mathbf{D}_s) \geq C_0 \big(\|\mathbf{E}_e\|^2 + \|\mathbf{D}_s\|^2 \big). \tag{36}$$

Then, the minimization problem (32)–(34) admits at least one minimizing solution pair $(\hat{\mathbf{y}}, \hat{\mathbf{R}}) \in \mathscr{A}$.

Proof. One can prove this statement using the direct methods of the calculus of variations. The procedure is very similar to the proof of Theorem 6 in [1], where we have formulated the same existence result, but expressed in terms of the shell bending-curvature tensor $\mathbf{K}_e$. By virtue of the extended Nye's formula (17) and the relation (21), we can adapt this proof to our case, i.e. when W is a function of $(\mathbf{E}_e, \mathbf{D}_s)$. For the sake of brevity, we shall present only the main steps of the proof and omit further detailed explanations.

We show first that the external loading potential satisfies the estimate

$$|\Lambda(\mathbf{y},\overline{\mathbf{R}})| \leq C\big(\|\mathbf{y}\|_{H^1(\omega)}+1\big), \qquad \forall(\mathbf{y},\overline{\mathbf{R}})\in\mathscr{A}.$$

where $C>0$ is a constant. Using this relation and the coercivity relation (36) we obtain

$$\mathscr{E}(\mathbf{y},\overline{\mathbf{R}}) \geq C_0\,\|\nabla\mathbf{y}\|^2_{L^2(\omega)} - C_1\|\,\mathbf{y}\,\|_{H^1(\omega)} - C_2\,,$$

where $C_1\,,C_2$ are some constants. To estimate the first term on the right-hand side of this inequality we apply the Poincaré–inequality and find (with $C_p>0$ constant)

$$\mathscr{E}(\mathbf{y},\overline{\mathbf{R}}) \geq C_p\,\|\mathbf{y}-\mathbf{y}^*\|^2_{H^1(\omega)} - C_3\|\mathbf{y}-\mathbf{y}^*\|_{H^1(\omega)} + C_4\,, \qquad \forall(\mathbf{y},\overline{\mathbf{R}})\in\mathscr{A}$$

so that the functional $\mathscr{E}(\mathbf{y},\overline{\mathbf{R}})$ is bounded from below over $\mathscr{A}$.

Then, there exists an infimizing sequence $(\mathbf{y}_n,\mathbf{R}_n)_{n\in\mathbf{N}}$ such that

$$\lim_{n\to\infty}\mathscr{E}(\mathbf{y}_n,\mathbf{R}_n) = \inf\big\{\mathscr{E}(\mathbf{y},\overline{\mathbf{R}})\,\big|\,(\mathbf{y},\overline{\mathbf{R}})\in\mathscr{A}\big\}. \tag{37}$$

We show that the sequences $(\mathbf{y}_n)$ and $(\mathbf{R}_n)$ are bounded in $H^1(\omega,\mathbb{R}^3)$ and $H^1(\omega,\mathbb{R}^{3\times3})$, respectively. Then, we can extract subsequences (not relabeled) such that

$$\begin{aligned}
&\mathbf{y}_n \rightharpoonup \hat{\mathbf{y}} \quad \text{in } H^1(\omega,\mathbb{R}^3) \qquad \text{and} \qquad \mathbf{y}_n \to \hat{\mathbf{y}} \quad \text{in } L^2(\omega,\mathbb{R}^3),\\
&\mathbf{R}_n \rightharpoonup \hat{\mathbf{R}} \quad \text{in } H^1(\omega,\mathbb{R}^{3\times3}) \qquad \text{and} \qquad \mathbf{R}_n \to \hat{\mathbf{R}} \quad \text{in } L^2(\omega,\mathbb{R}^{3\times3}).
\end{aligned}$$

The limit elements satisfy $(\hat{\mathbf{y}},\hat{\mathbf{R}})\in\mathscr{A}$ and we can construct the corresponding shell strain measures $\hat{\mathbf{E}}_e\,,\hat{\mathbf{D}}_s$, as well as $(\mathbf{E}_e)_n\,,(\mathbf{D}_s)_n$, using the definitions (6) and (16). Then, we show the weak convergence (on subsequences)

$$(\mathbf{E}_e)_n \rightharpoonup \hat{\mathbf{E}}_e \quad \text{in } L^2(\omega,\mathbb{R}^{3\times3}) \qquad \text{and} \qquad (\mathbf{D}_s)_n \rightharpoonup \hat{\mathbf{D}}_s \quad \text{in } L^2(\omega,\mathbb{R}^{3\times3}).$$

Finally, we use the convexity of the strain energy function $\widehat{W}(\mathbf{E}_e,\mathbf{D}_s)$ and deduce

$$\int_\omega \widehat{W}(\hat{\mathbf{E}}_e,\hat{\mathbf{D}}_s)\,a(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 \;\leq\; \liminf_{n\to\infty}\int_\omega \widehat{W}\big((\mathbf{E}_e)_n,(\mathbf{D}_s)_n\big)\,a(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\,.$$

Fron this inequality and (33) it follows that :

$$\mathscr{E}(\hat{\mathbf{y}},\hat{\mathbf{R}}) \;\leq\; \liminf_{n\to\infty}\,\mathscr{E}(\mathbf{y}_n,\mathbf{R}_n). \tag{38}$$

In view of (37) and (38) we see that $(\hat{\mathbf{y}},\hat{\mathbf{R}})$ is a minimizing solution pair of our minimization problem (32)–(34). □

5 Application: isotropic Cosserat shells

We employ the above results to investigate the case of isotropic elastic shells. To write the specific form of the strain energy density for isotropic shells, we use the decomposition into the *planar part* and *out-of-plane part* of the shell strain tensors $\mathbf{E}_e$ and $\mathbf{K}_e$:

$$\begin{aligned}\mathbf{E}_e = (\mathbf{a}+\mathbf{n}_0\otimes\mathbf{n}_0)\mathbf{E}_e = \mathbf{a}\mathbf{E}_e + \mathbf{n}_0\otimes(\mathbf{n}_0\mathbf{E}_e), \quad \|\mathbf{E}_e\|^2 = \|\mathbf{a}\mathbf{E}_e\|^2 + \|\mathbf{n}_0\mathbf{E}_e\|^2,\\ \mathbf{K}_e = \mathbf{a}\mathbf{K}_e + \mathbf{n}_0\otimes(\mathbf{n}_0\mathbf{K}_e), \qquad \|\mathbf{K}_e\|^2 = \|\mathbf{a}\mathbf{K}_e\|^2 + \|\mathbf{n}_0\mathbf{K}_e\|^2.\end{aligned} \tag{39}$$

In [8] the following general form of the strain energy density $W(\mathbf{E}_e,\mathbf{K}_e)$ for 6-parameter isotropic elastic shells was proposed

$$\begin{aligned}2W(\mathbf{E}_e,\mathbf{K}_e) = \alpha_1\big[\mathrm{tr}(\mathbf{a}\mathbf{E}_e)\big]^2 + \alpha_2\,\mathrm{tr}(\mathbf{a}\mathbf{E}_e)^2 + \alpha_3\,\|\mathbf{a}\mathbf{E}_e\|^2 + \alpha_4\,\|\mathbf{n}_0\mathbf{E}_e\|^2\\ +\beta_1\big[\mathrm{tr}(\mathbf{a}\mathbf{K}_e)\big]^2 + \beta_2\,\mathrm{tr}(\mathbf{a}\mathbf{K}_e)^2 + \beta_3\,\|\mathbf{a}\mathbf{K}_e\|^2 + \beta_4\,\|\mathbf{n}_0\mathbf{K}_e\|^2,\end{aligned} \tag{40}$$

where α_k and β_k are constant constitutive coefficients ($k = 1,2,3,4$).

We want to express the strain energy density as a function $\widehat{W}(\mathbf{E}_e,\mathbf{D}_s)$. To this aim, we decompose the shell dislocation density tensor $\mathbf{D}_s$ as

$$\mathbf{D}_s = \mathbf{D}_s(\mathbf{a}+\mathbf{n}_0\otimes\mathbf{n}_0) = \mathbf{D}_s\,\mathbf{a} + (\mathbf{D}_s\,\mathbf{n}_0)\otimes\mathbf{n}_0, \qquad \|\mathbf{D}_s\|^2 = \|\mathbf{D}_s\,\mathbf{a}\|^2 + \|\mathbf{D}_s\,\mathbf{n}_0\|^2$$

and employ the extended Nye's formula $(17)_2$ to write

$$\mathbf{a}\mathbf{K}_e = -(\mathbf{D}_s\,\mathbf{a})^T + \mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big)\,\mathbf{a}, \qquad \mathbf{n}_0\mathbf{K}_e = -\mathbf{n}_0\mathbf{D}_s^T = -\mathbf{D}_s\,\mathbf{n}_0\,. \tag{41}$$

From (41) it follows

$$\mathrm{tr}\big(\mathbf{a}\mathbf{K}_e\big) = -\mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big)^T + 2\,\mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big) = \mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big), \qquad \|\mathbf{n}_0\mathbf{K}_e\| = \|\mathbf{D}_s\,\mathbf{n}_0\|\,. \tag{42}$$

In view of (21), $(39)_4$ and $(42)_2$ we get

$$\|\mathbf{a}\mathbf{K}_e\|^2 = \|\mathbf{K}_e\|^2 - \|\mathbf{n}_0\mathbf{K}_e\|^2 = \|\mathbf{D}_s\|^2 - \|\mathbf{D}_s\,\mathbf{n}_0\|^2 = \|\mathbf{D}_s\,\mathbf{a}\|^2. \tag{43}$$

Furthermore, from $(41)_1$ we deduce

$$(\mathbf{a}\mathbf{K}_e)^2 = (\mathbf{D}_s\,\mathbf{a})^T(\mathbf{D}_s\,\mathbf{a})^T - 2\big[\mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big)\big](\mathbf{D}_s\,\mathbf{a})^T + \big[\mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big)\big]^2\mathbf{a}$$

and applying the trace operator we find

$$\mathrm{tr}(\mathbf{a}\mathbf{K}_e)^2 = \mathrm{tr}(\mathbf{D}_s\,\mathbf{a})^2 - 2\big[\mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big)\big]^2 + 2\big[\mathrm{tr}\big(\mathbf{D}_s\,\mathbf{a}\big)\big]^2 = \mathrm{tr}(\mathbf{D}_s\,\mathbf{a})^2. \tag{44}$$

If we insert the relations (42)–(44) into (40) we obtain the following expression of the strain energy density in terms of the shell strain tensor $\mathbf{E}_e$ and the shell dislocation density tensor $\mathbf{D}_s$

$$\begin{aligned}2\widehat{W}(\mathbf{E}_e,\mathbf{D}_s) = \;&\alpha_1\big[\mathrm{tr}(\mathbf{a}\mathbf{E}_e)\big]^2 + \alpha_2\,\mathrm{tr}(\mathbf{a}\mathbf{E}_e)^2 + \alpha_3\,\|\mathbf{a}\mathbf{E}_e\|^2 + \alpha_4\,\|\mathbf{n}_0\mathbf{E}_e\|^2\\ &+\beta_1\big[\mathrm{tr}(\mathbf{D}_s\,\mathbf{a})\big]^2 + \beta_2\,\mathrm{tr}(\mathbf{D}_s\,\mathbf{a})^2 + \beta_3\,\|\mathbf{D}_s\,\mathbf{a}\|^2 + \beta_4\,\|\mathbf{D}_s\,\mathbf{n}_0\|^2.\end{aligned} \tag{45}$$

We can put this expression in a more convenient form. Using the orthogonal decomposition of the type (18) we derive

$$\|\,\mathbf{D}_s\,\mathbf{a}\|^2 = \|\mathrm{dev}_s\mathrm{sym}\,(\mathbf{D}_s\,\mathbf{a})\|^2 + \|\mathrm{skew}\,(\mathbf{D}_s\,\mathbf{a})\|^2 + \frac{1}{2}\left[\mathrm{tr}\,(\mathbf{D}_s\,\mathbf{a})\right]^2 \tag{46}$$

and

$$\mathrm{tr}(\mathbf{D}_s\,\mathbf{a})^2 = \|\mathrm{dev}_s\mathrm{sym}\,(\mathbf{D}_s\,\mathbf{a})\|^2 - \|\mathrm{skew}\,(\mathbf{D}_s\,\mathbf{a})\|^2 + \frac{1}{2}\left[\mathrm{tr}\,(\mathbf{D}_s\,\mathbf{a})\right]^2. \tag{47}$$

Similar expressions can be obtained for $\|\mathbf{aE}_e\|^2$ and $\mathrm{tr}(\mathbf{aE}_e)^2$. In view of relations (46) and (47) we can finally write the strain energy density (45) in the form

$$\begin{aligned} 2\,\widehat{W}(\mathbf{E}_e,\mathbf{D}_s) =\, & (\alpha_2+\alpha_3)\|\mathrm{dev}_s\mathrm{sym}\,(\mathbf{aE}_e)\|^2 + (\alpha_3-\alpha_2)\|\mathrm{skew}\,(\mathbf{aE}_e)\|^2 \\ & +\big(\alpha_1+\frac{\alpha_2+\alpha_3}{2}\big)\left[\mathrm{tr}\,(\mathbf{aE}_e)\right]^2 + \alpha_4\|\mathbf{n}_0\mathbf{E}_e\|^2 \\ & +(\beta_2+\beta_3)\|\mathrm{dev}_s\mathrm{sym}\,(\mathbf{D}_s\,\mathbf{a})\|^2 + (\beta_3-\beta_2)\|\mathrm{skew}\,(\mathbf{D}_s\,\mathbf{a})\|^2 \\ & +\big(\beta_1+\frac{\beta_2+\beta_3}{2}\big)\left[\mathrm{tr}\,(\mathbf{D}_s\,\mathbf{a})\right]^2 + \beta_4\|\mathbf{D}_s\,\mathbf{n}_0\|^2. \end{aligned} \tag{48}$$

By virtue of Lemma 1 we see that the quadratic function (48) is coercive if and only if the constitutive coefficients satisfy the inequalities

$$\begin{aligned} &\alpha_1+\frac{\alpha_2+\alpha_3}{2}>0, \qquad \alpha_2+\alpha_3>0, \qquad \alpha_3-\alpha_2>0, \qquad \alpha_4>0, \\ &\beta_1+\frac{\beta_2+\beta_3}{2}>0, \qquad \beta_2+\beta_3>0, \qquad \beta_3-\beta_2>0, \qquad \beta_4>0. \end{aligned} \tag{49}$$

Under these conditions, all the hypotheses on the strain energy density W required by Theorem 1 are fulfilled. Thus, we can apply the Theorem 1 to prove the existence of minimizers for isotropic Cosserat shells.

Remark 4. To apply the model in practical situations it is useful to express the constitutive coefficients α_k and β_k in terms of the material parameters of the elastic continuum and the thickness of the shell. In the case of an isotropic Cauchy elastic material (characterized by the Young modulus E and the Poisson ratio ν), a particular (simplified) expression of the strain energy density $W(\mathbf{E}_e,\mathbf{K}_e)$ has been proposed in [4, 5]

$$\begin{aligned} 2\,W(\mathbf{E}_e,\mathbf{K}_e) = C\nu\left[\mathrm{tr}(\mathbf{aE}_e)\right]^2 + C(1-\nu)\,\|\,\mathbf{aE}_e\|^2 + \kappa_s C(1-\nu)\,\|\,\mathbf{n}_0\mathbf{E}_e\|^2 \\ +D\nu\left[\mathrm{tr}(\mathbf{aK}_e)\right]^2 + D(1-\nu)\,\|\,\mathbf{aK}_e\|^2 + \kappa_t\,D(1-\nu)\,\|\,\mathbf{n}_0\mathbf{K}_e\|^2, \end{aligned} \tag{50}$$

where $C=\frac{E\,h}{1-\nu^2}$ is the stretching (membrane) stiffness of the shell, $D=\frac{E\,h^3}{12(1-\nu^2)}$ is the bending stiffness, h is the thickness of the shell, and κ_s, κ_t are two shear correction factors. In [5], the values of the shear correction factors have been set to $\alpha_s=\frac{5}{6}$, $\alpha_t=\frac{7}{10}$ using the numerical treatment of some non-linear shell problems.

The same values have been also proposed previously in the literature, see e.g. [18, 9, 15].

In view of (42), (43), we can write the function (50) in terms of the shell dislocation density tensor $\mathbf{D}_s$

$$
\begin{aligned}
2\widehat{W}(\mathbf{E}_e,\mathbf{D}_s) = {} & C\nu\big[\mathrm{tr}(\mathbf{a}\mathbf{E}_e)\big]^2 + C(1-\nu)\,\|\,\mathbf{a}\mathbf{E}_e\|^2 + \kappa_s C(1-\nu)\,\|\,\mathbf{n}_0\mathbf{E}_e\|^2 \\
& + D\nu\big[\mathrm{tr}(\mathbf{D}_s\,\mathbf{a})\big]^2 + D(1-\nu)\,\|\,\mathbf{D}_s\,\mathbf{a}\|^2 + \kappa_t\, D(1-\nu)\,\|\,\mathbf{D}_s\,\mathbf{n}_0\|^2 .
\end{aligned} \tag{51}
$$

This corresponds to the following choice of the constitutive coefficients α_k and β_k

$$
\begin{aligned}
&\alpha_1 = C\,\nu, \qquad \alpha_2 = 0, \qquad \alpha_3 = C\,(1-\nu), \qquad \alpha_4 = \kappa_s C\,(1-\nu), \\
&\beta_1 = D\,\nu, \qquad \beta_2 = 0, \qquad \beta_3 = D\,(1-\nu), \qquad \beta_4 = \kappa_t\, D\,(1-\nu).
\end{aligned} \tag{52}
$$

To verify the conditions (49) we compute using (52)

$$
\begin{aligned}
&\alpha_1 + \frac{\alpha_2+\alpha_3}{2} = C\cdot\frac{1+\nu}{2} = h\cdot\frac{E}{2(1-\nu)} = h\cdot\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}\,, \\
&\alpha_2+\alpha_3 = \alpha_3-\alpha_2 = h\cdot\frac{E}{1+\nu} = 2h\,\mu, \qquad \alpha_4 = 2h\,\kappa_s\,\mu, \\
&\beta_1 + \frac{\beta_2+\beta_3}{2} = D\cdot\frac{1+\nu}{2} = \frac{h^3}{24}\cdot\frac{E}{1-\nu} = \frac{h^3}{12}\cdot\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}\,, \\
&\beta_2+\beta_3 = \beta_3-\beta_2 = \frac{h^3}{12}\cdot\frac{E}{1+\nu} = \frac{h^3}{6}\cdot\mu, \qquad \beta_4 = \frac{h^3}{6}\cdot\kappa_t\,\mu,
\end{aligned} \tag{53}
$$

where λ and μ are the Lamé constants of the isotropic and homogeneous elastic material. If we insert the relations (53) into the general expression (48), then we obtain the appropriate form of the strain energy density in this model

$$
\begin{aligned}
\widehat{W}(\mathbf{E}_e,\mathbf{D}_s) = {} & \mu\,h\,\Big[\|\mathrm{dev}_s\mathrm{sym}\,(\mathbf{a}\mathbf{E}_e)\|^2 + \|\mathrm{skew}\,(\mathbf{a}\mathbf{E}_e)\|^2 + \frac{3\lambda+2\mu}{2(\lambda+2\mu)}\,\big[\mathrm{tr}\,(\mathbf{a}\mathbf{E}_e)\big]^2 \\
& + \kappa_s\,\|\mathbf{n}_0\mathbf{E}_e\|^2\Big] + \mu\,\frac{h^3}{12}\,\Big[\|\mathrm{dev}_s\mathrm{sym}\,(\mathbf{D}_s\,\mathbf{a})\|^2 + \|\mathrm{skew}\,(\mathbf{D}_s\,\mathbf{a})\|^2 \\
& + \frac{3\lambda+2\mu}{2(\lambda+2\mu)}\,\big[\mathrm{tr}\,(\mathbf{D}_s\,\mathbf{a})\big]^2 + \kappa_t\,\|\mathbf{D}_s\,\mathbf{n}_0\|^2\Big].
\end{aligned} \tag{54}
$$

We see that the inequalities (49) are satisfied in this case, provided

$$
E > 0, \qquad -1 < \nu < \frac{1}{2}\,,
$$

or equivalently, in terms of the Lamé constants,

$$
\mu > 0, \qquad 3\lambda + 2\mu > 0.
$$

These inequalities are satisfied, in view of the positive definiteness of the three-dimensional quadratic elastic strain energy for isotropic materials.

Since the conditions (49) hold, we are able to apply the Theorem 1 and we obtain the existence of minimizers also for this special constitutive model.

6 Open problems

The presented existence result based on strict convexity in the employed strain and curvature measures does not exhaust all possibilities. Indeed, from a modelling point of view it is pertinent to consider a generalized stress-strain relation for which only an estimate of the elastic energy in terms of symmetrized elastic strains is available, i.e. only an estimate of the type

$$\widehat{W}(\mathbf{E}_e,\mathbf{D}_s) \geq C\left(\|\mathrm{sym}\,\mathbf{E}_e\|^2+\|\mathbf{D}_s\|^2\right) \tag{55}$$

is available (instead of (36)). Whether this situation remains well-posed is presently not known.

In our example for isotropic Cosserat shells, this case corresponds to $\alpha_2=\alpha_3$, since the contribution of $\|\mathrm{skew}\,\mathbf{E}_e\|^2$ in (48) would then vanish. If we consider isotropic elastic shells made of an Cosserat material [10], this situation corresponds to the case $\mu_c=0$, where μ_c denotes the Cosserat couple modulus of the three-dimensional continuum [12]. Similar problems have been dealt with in [13], where the case of vanishing Cosserat couple modulus has been investigated.

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