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# Curl bounds Grad on $\mathrm{SO}(3)$

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Let $F^{\mathrm{p}} \in \mathrm{GL}(3)$ be the plastic deformation from the multiplicative decomposition in elasto-plasticity. We show that the geometric dislocation density tensor of Gurtin in the form $\mathrm{Curl}[F^{\mathrm{p}}] \cdot (F^{\mathrm{p}})^T$ applied to rotations controls the gradient in the sense that pointwise

$$\forall\, R \in C^1(\mathbb{R}^3, \mathrm{SO}(3)) : \quad \|\mathrm{Curl}[R] \cdot R^T\|^2_{\mathbb{M}^{3\times3}} \geq \frac{1}{2}\, \|\mathrm{D}R\|^2_{\mathbb{R}^{27}} .$$

This result complements rigidity results (John, Reshetnyak, Friesecke/James/Müller) as well as an associated linearized theorem saying that

$$\forall\, A \in C^1(\mathbb{R}^3, \mathrm{so}(3)) : \quad \|\mathrm{Curl}[A]\|^2_{\mathbb{M}^{3\times3}} \geq \frac{1}{2}\, \|\mathrm{D}A\|^2_{\mathbb{R}^{27}} = \|\nabla \mathrm{axl}[A]\|^2_{\mathbb{R}^9} .$$



## 1 Introduction

We show an extension to the Lie-group $\mathrm{SO}(3)$ of proper rotations of the following result for linearized kinematics: the operator $\mathrm{Curl}_\sharp$ (curl arranged row wise) applied to elements of the Lie-algebra of skew-symmetric matrices $\mathrm{so}(3)$ already controls all partial derivatives of these matrices. While in general, the operator $\mathrm{Curl}_\sharp$ cannot control the full gradient since $\mathrm{Curl}_\sharp$ has 9 independent entries but $\mathrm{Grad} = \mathrm{D}$ has 27 independent entries, it does so on $\mathrm{so}(3)$, since they have only 3 independent components such that taking Grad gives 9 independent entries making the relation between $\mathrm{Curl}_\sharp$ and Grad invertible.

Such a result can at least be traced back implicitly to Nye [1], who investigated infinitesimal rotations of the crystal lattice due to dislocation motion. He showed for small plastic deformations and zero elastic strains that

$$-\mathrm{Curl}_\sharp[\mathrm{skew}[\varepsilon^{\mathrm{p}}]] = (\nabla \mathrm{axl}[\mathrm{skew}[\varepsilon^{\mathrm{p}}]])^T - \mathrm{tr}[(\nabla \mathrm{axl}[\mathrm{skew}[\varepsilon^{\mathrm{p}}]])^T]\, 1\!\!1 , \tag{1}$$

where $\varepsilon^{\mathrm{p}} \in C^1(\Omega, \mathbb{M}^{3\times3})$ is the *non-symmetrical infinitesimal plastic distortion* with $\Omega \subset \mathbb{R}^3$ the reference configuration. Here, for second order tensors $\mathrm{skew}[X] := \frac{1}{2}(X - X^T)$, $1\!\!1$ is the identity tensor, $\|X\|^2 = \sum_i X_i^2$, $\mathrm{tr}[X]$ the trace, the axial vector $\mathrm{axl}[A]$ is defined such that $A \cdot v = \mathrm{axl}[A] \times v$ for all $A \in \mathrm{so}(3)$ and $v \in \mathbb{R}^3$ and $\nabla\varphi$ is the Jacobian-matrix. With $A \cdot B$ we denote simple contraction, with $A : B$ double contraction. Abbreviating $A = \mathrm{skew}[\varepsilon^{\mathrm{p}}] \in C^1(\Omega, \mathrm{so}(3))$ one deduces

$$-\mathrm{Curl}_\sharp[A] = (\nabla \mathrm{axl}[A])^T - \mathrm{tr}[(\nabla \mathrm{axl}[A])^T]1\!\!1 \;\Leftrightarrow\; \nabla \mathrm{axl}[A] = -(\mathrm{Curl}_\sharp[A])^T + \frac{1}{2}\mathrm{tr}[(\mathrm{Curl}_\sharp[A])^T]1\!\!1 , \tag{2}$$

which implies

$$\forall\, A \in C^1(\mathbb{R}^3, \mathrm{so}(3)) : \quad \|\mathrm{Curl}_\sharp[A]\|^2_{\mathbb{M}^{3\times3}} \geq \frac{1}{2}\, \|\mathrm{D}A\|^2_{\mathbb{R}^{27}} = \|\nabla \mathrm{axl}[A]\|^2_{\mathbb{R}^9} , \tag{3}$$

in turn implying infinitesimal rigidity (7). Recall also the definition of the curl of displacements $u \in C^1(\Omega, \mathbb{R}^3)$ and the relation to the infinitesimal rotations $\mathrm{skew}[\nabla u]$,

$$\mathrm{curl}[u] := \nabla \times u = 2\, \mathrm{axl}[\mathrm{skew}[\nabla u]] . \tag{4}$$

The modern theory of finite plasticity is based on the multiplicative decomposition $F = F^{\mathrm{e}} \cdot F^{\mathrm{p}}$ of the deformation gradient $F = \nabla\varphi$ into structural elastic and plastic components. In single crystal plasticity $F^{\mathrm{p}}$ represents the deformation solely resulting from the formation of defects such as dislocations while $F^{\mathrm{e}}$ is due to elastic stretch and elastic rotation of the lattice. In general, $F^{\mathrm{e}}$ and $F^{\mathrm{p}}$ have not the form of a Jacobian matrix, they are incompatible, i.e.$\mathrm{Curl}_\sharp[F^{\mathrm{e}}], \mathrm{Curl}_\sharp[F^{\mathrm{p}}] \neq 0$, a property related to the formation of dislocations. The most general stored defect energy, measuring the incompatibility in $F^{\mathrm{p}}$, which is invariant under a compatible change in the reference configuration is expressible in the *geometrical dislocation density tensor* $G = \frac{1}{\det[F^{\mathrm{p}}]}\mathrm{Curl}_\sharp[F^{\mathrm{p}}] \cdot (F^{\mathrm{p}})^T$ which, for $R \in \mathrm{SO}(3)$, reduces to $G = \mathrm{Curl}_\sharp[R] \cdot R^T$. For the necessary background and

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more references on dislocations, plasticity and microstructures we refer to [5].

Another motivation comes from rigidity results [2, 3] in the spirit of Liouville-type theorems, saying that if the gradient of a deformation is locally a rotation it must be a constant rotation together with a constant translation or more precisely

$$\varphi \in W^{1,\infty}(\Omega,\mathbb{R}^n), \quad \nabla\varphi(x) \in \mathrm{SO}(n)\ a.e \ \Rightarrow \\ \nabla\varphi = R = \text{const.} \Leftrightarrow \varphi(x) = R\cdot x + b\,. \tag{5}$$

A quantized version of this fact has been given recently in [4]. They show that for bounded $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary and $\varphi \in W^{1,2}(\Omega,\mathbb{R}^3)$ there exists a positive constant $C(\Omega)$ and a constant rotation $R$ such that

$$\int_\Omega \|\nabla\varphi - R\|^2\,\mathrm{dx} \le C(\Omega)\int_\Omega \mathrm{dist}^2(\nabla\varphi,\mathrm{SO}(3))\,\mathrm{dx}\,. \tag{6}$$

The respective infinitesimal rigidity result is standard in the treatment of linear elasticity and Korn's inequality. It amounts to

$$u \in W^{1,2}(\Omega,\mathbb{R}^3)\,, \quad \nabla u(x) + \nabla u(x)^T = 0 \ \Leftrightarrow\ \nabla u(x) \in \mathrm{so}(3) \ \Leftrightarrow \\ \nabla u(x) = A = \text{const.} \Leftrightarrow u(x) = A\cdot x + b\,, \tag{7}$$

where $A \in \mathrm{so}(3)$ and $b \in \mathbb{R}^3$ are constant. Since from $\mathrm{sym}[\nabla u(x)] = 0$ it follows $\nabla u(x) = A(x) \in \mathrm{so}(3)$ the result (7) would follow by applying $\mathrm{Curl}_\sharp$ on both sides and using that $\mathrm{Curl}_\sharp$ bounds $\mathrm{D}A$ on $\mathrm{so}(3)$ due to (3).

As a consequence of (5) it is known that for smooth, simply connected domains $\Omega \subset \mathbb{R}^3$ and $R \in C^1(\Omega,\mathrm{SO}(3))$

$$0 = \mathrm{Curl}_\sharp[R(x)] \ \Leftrightarrow\ R = \nabla\varphi \in \mathrm{SO}(3) = \text{const.} \Leftrightarrow \mathrm{D}R = 0\,, \tag{8}$$

thus showing that $\mathrm{Curl}_\sharp[R] = 0 \Leftrightarrow \mathrm{D}R = 0$. Obviously, $\|\mathrm{Curl}_\sharp[R]\|^2_{\mathbb{M}^{3\times3}} \le 2\,\|\mathrm{D}R\|^2_{\mathbb{R}^{27}}$ by Young's inequality for all $R \in \mathbb{M}^{3\times3}$. The precise relation between $\mathrm{Curl}_\sharp$ and $\mathrm{Grad} = \mathrm{D}$ on $\mathrm{SO}(2)$ is easily understood in terms of the representation with one rotation angle $\vartheta : \Omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$

$$R(x,y) = \begin{pmatrix} \cos\vartheta(x,y) & \sin\vartheta(x,y) \\ -\sin\vartheta(x,y) & \cos\vartheta(x,y)\end{pmatrix} \in \mathrm{SO}(2)\,. \tag{9}$$

One checks that

$$\|\mathrm{Curl}_\sharp[R(x,y)]\|^2_{\mathbb{R}^2} = ((\cos\vartheta)_x - (\sin\vartheta)_y)^2 + ((-\sin\vartheta)_y - (\cos\vartheta)_x)^2 \\ = \|\nabla\vartheta(x,y)\|^2_{\mathbb{R}^2} = \frac{1}{2}\|\mathrm{D}R\|^2_{\mathbb{R}^8}\,, \tag{10}$$

which led us to surmise that for three-space dimensions

$$\exists\, c^+ > 0 \quad \forall\, R \in C^1(\mathbb{R}^3,\mathrm{SO}(3)) : \quad \|\mathrm{Curl}_\sharp[R]\|^2_{\mathbb{M}^{3\times3}} \ge c^+\,\|\mathrm{D}R\|^2_{\mathbb{R}^{27}}\,. \tag{11}$$

This is true for $c = \frac{1}{2}$. In terms of the *geometrical dislocation density tensor* $G = \mathrm{Curl}_\sharp[R]\cdot R^T$ we observe that $\|\mathrm{Curl}_\sharp[R]\|^2_{\mathbb{M}^{3\times3}} = \|\mathrm{Curl}_\sharp[R]\cdot R^T\|^2_{\mathbb{M}^{3\times3}}$ by the invariance of the euclidean norm under $\mathrm{SO}(3)$. The non-trivial implication in (5) is a simple consequence of (11). For the proof we refer to [6].

## References


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