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Existence results in dislocation based rate-independent isotropic gradient plasticity with kinematical hardening and plastic spin: The case with symmetric local backstress

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Abstract

In this paper we use convex analysis and variational inequality methods to establish an existence result for a model of infinitesimal rate-independent gradient plasticity with kinematic hardening and plastic spin, in which the local backstress tensor remains symmetric. The model features a defect energy contribution which is quadratic in the dislocation density tensor Curl p, giving rise to nonlocal non-symmetric kinematic hardening. Use is made of a recently established Korn's type inequality for incompatible tensor fields. The solution space for the non-symmetric plastic distortion is naturally H(Curl) together with suitable tangential boundary conditions on the plastic distortion. Connections to other models are established as well.

Key words: plasticity, gradient plasticity, dislocations, plastic spin, Korn's inequality, incompatible distortions, rate-independent models, kinematical hardening, backstress, variational inequality, defect energy.

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Contents

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1 Introduction

In the past twenty years, there have been several experimental investigations for metallic and ceramic materials which show that the elastic-plastic deformation of those materials are size-dependent for sufficiently small scales. This phenomenon cannot be predicted by conventional theories of plasticity, which do not include any material length scales. Hence, there clearly appeared a gap between micro-mechanical plasticity and classical continuum plasticity. The purpose of the enhanced gradient plasticity theories is to formulate a constitutive framework on the continuum level which is used to bridge that gap. In this paper we will only discuss phenomenological models of isotropic polycrystalline plasticity excluding the important case of single crystal plasticity.

There is now an abundant literature, and research activities towards the development of models of gradient plasticity which capture better the observed size-dependency mentioned above. In the works of Mühlhaus and Aifantis [33] and Gudmundson [20], the yield-stress is set to depend also on some derivative of a scalar measure of the accumulated plastic distortion. In the works of Gurtin and Anand [21], Gudmundson [20] and Neff et al. [34], the yield-stress is not modified, but the free-energy is augmented by a term involving the dislocation density. Also, it is assumed in [21] that the plastic flow is governed not necessarily by the stress deviator (as in classical plasticity), but more generally by microstress tensors that also satisfy a balance law.

In [33, 20, 21], the plastic distortion variable is assumed to be symmetric. However, as Gurtin and Anand note [21, p. 1626]: "…unless the plastic spin is (explicitly) constrained to zero, constitutive dependencies on the Burger tensor necessarily involve dependencies on the (infinitesimal) plastic rotation". Note that even in classical plasticity, the effect of plastic spin has been studied by several authors like Dafalias [11, 12], Mandel [28, 29] and Kratochvil [25, 26], who were the first to suggest that a complete macroscopic plasticity theory must include constitutive relations involving also the plastic spin.

On the other hand, though there are several theories of gradient plasticity available in the literature, the results of mathematical analysis for these problems are still rather scarce. The first result of mathematical analysis on a model of gradient plasticity was due to Djoko et al. [14]. While the developments by Gurtin-Anand [21] were done for viscoplastic bodies, the well-posedness of that model is considered by Reddy at al. [46] for the rate-independent problem with isotropic hardening and with both energetic and dissipative length scales involved. Also, computational aspects of the model, based on the work [23], are studied in [7] and are devoted exclusively to single crystal plasticity. Let us mention that, the purely energetic version of the Gurtin-Anand model, i.e., when the dissipative length scale $\ell = 0$, is not yet treated but will be done in this paper via the identification with the irrotational version of Ebobisse-Neff [17] presented in Paragraph 3.4. We also study in Section 5, the purely energetic case of the Gurtin-Anand model with linear kinematic hardening. Another existence result for the rate-independent problem of the Gurtin-Anand model was obtained by Giaccomini and Lussardi [19] within the energetic-approach developed by Mielke [31, 32] and it has also been proved that the model converges in a suitable sense to a formulation of classical perfect plasticity proposed in [13] whenever the energetic and dissipative length scales go to zero.

Neff et al. proposed in [34] a model of finite strain gradient plasticity based on the multiplicative decomposition including phenomenological Prager type symmetric linear kinematical hardening and nonlocal kinematical hardening due to dislocations. The model is from the outset non spin-free (the plastic distortion p is not symmetric) and its linearization leads to a thermodynamically admissible model of infinitesimal plasticity involving only the Curl of the non-symmetric plastic distortion p. The well-posedness of the linear model is addressed as well, when formulated as a variational inequality.

In [17], we have studied the well-posedness of this model described within the framework of the dual formulation for isotropic hardening by

Additive split of distortion:	$\nabla u = e + p$
Additive split of strain:	$\mathrm{sym}\,\nabla u = \varepsilon = \varepsilon^e + \varepsilon^p\,, \quad \varepsilon^e = \mathrm{sym}\; e$
Equilibrium:	$\mathrm{Div}\,\sigma + f = 0$ with $\sigma = \mathbb{C}.\varepsilon^e$
Dissipation inequality:	$\int_\Omega [\langle \sigma - \mu\, L_c^2 \;\mathrm{Curl\; Curl}\; p, \dot p\rangle - k_2\,\gamma\,\dot\gamma]\,dx \geq 0$
Flow law in dual form:	$\dot p \in \partial\chi(\sigma - \mu\, L_c^2\,\mathrm{Curl\,Curl}\,p), \qquad \dot\gamma = \lvert\dot p\rvert$ χ is the indicator function of the set of admissible stresses

Table 1: The model with plastic spin and isotropic hardening in [17].

In this paper, we settle some of the open questions raised in [17]. In that paper, we dealt with the isotropic hardening case only and here we would like to extend our analysis to the local linear Prager type kinematical hardening model which, in the dual formulation of classical plasticity has the flow law

$$\dot\varepsilon^p \in \partial\chi(\sigma - b) \quad \text{ and } \quad \dot b = \mu\, k_1\, \dot\varepsilon^p\,, \tag{1.1}$$

in which b is the symmetric backstress tensor. Notice that $(1.1)_2$ can be explicitly integrated (with proper initial conditions $\varepsilon^p(0) = 0$, $b(0) = 0$) to yield $b = \mu\, k_1\, \varepsilon^p$ and, substituted in $(1.1)_1$, gives

$$\dot{\varepsilon}^p \in \partial\chi(\sigma - \mu\, k_1\, \varepsilon^p)\,.$$

Applying the same reasoning to our nonlocal gradient model, we want to add a local backstress similar to $(1.1)_2$. Hence, we define

$$\dot{b} = \mu\, k_1\, \mathrm{sym}\, \dot{p}\,,$$

in which we conserve the symmetry of the local backstress tensor b. So, we get similarly,

$$\dot{p} \in \partial\chi(\sigma - b - \mu L_c^2\, \mathrm{Curl}\, \mathrm{Curl}\, p) \quad \text{and} \quad \dot{b} = \mu\, k_1\, \mathrm{sym}\, \dot{p}$$

and integrating as above yields

$$\dot{p} \in \partial\chi(\Sigma_E) \subset \mathbb{R}^+ \frac{\mathrm{dev}\ \Sigma_E}{|\,\mathrm{dev}\ \Sigma_E|} \quad \text{with} \quad \Sigma_E = \sigma - \mu\, k_1\, \mathrm{sym}\, p - \mu\, L_c^2\, \mathrm{Curl}\, \mathrm{Curl}\, p\,. \tag{1.2}$$

Here, Σ_E is the elastic Eshelby tensor driving the plastic evolution. Note immediately that it is only the nonlocal backstress contribution $\mu\, L_c^2$ Curl Curl p which is responsible for the appearance of plastic spin or not. In order to substantiate this claim, set $L_c = 0$ in $(1.2)_2$ and consider

$$\dot{p} = \lambda\, \frac{\mathrm{dev}\ \Sigma_E}{|\,\mathrm{dev}\ \Sigma_E|} \in \mathrm{Sym}(3)\,.$$

Assuming $p(0) \in \mathrm{Sym}(3)$, we get that $p(t) \in \mathrm{Sym}(3)$ for every t and hence we may replace p with $\varepsilon^p := \mathrm{sym}\ p$.

From a purely mathematical point of view, we could also consider the case of a non-symmetric local backstress tensor, i.e.,

$$\dot{p} \in \partial\chi(\sigma - \widehat{b} - \mu\, L_c^2\, \mathrm{Curl}\, \mathrm{Curl}\, p)\,, \quad \text{and} \quad \dot{\widehat{b}} = \mu\, k_1\, \dot{p}\,,$$

which integrates to

$$\dot{p} \in \partial\chi(\sigma - \mu\, k_1\, p - \mu\, L_c^2\, \mathrm{Curl}\, \mathrm{Curl}\, p)\,. \tag{1.3}$$

In this case, the following mathematical analysis would be drastically simplified since no new estimate of the Korn type on incompatible tensor fields (see [40, 37, 38, 39]) is needed. The solution space is then trivially H(Curl). Notice that (1.3) also still reduces to a formulation of classical plasticity in terms of a symmetric plastic strain tensor ε^p if the energetic length scale L_c vanishes and the initial plastic distortion $p(0)$ is chosen to be symmetric. To see this, consider for $L_c = 0$, the equation

$$\dot{p} = \lambda\, \frac{\mathrm{dev}\ \sigma - \mu\, k_1\, p}{|\,\mathrm{dev}\ \sigma - \mu\, k_1\, p|}\,.$$

The format of the equation, as far as classical solutions is concerned, is of the type

$$\dot{p} = S(t) - \alpha\, p(t), \quad p(0) = p_0 \in \mathrm{Sym}(3)\,, \tag{1.4}$$

where $S(t) \in \mathrm{Sym}(3)$ and $\alpha \in \mathbb{R}$ can be assumed given. Clearly, the system (1.4) has only symmetric solutions $p(t)$.
The total energy would be of the type (isotropic elastic response for simplicity)

$$\mu\,|\mathrm{sym}\,(\nabla u - p)|^2 + \frac{\lambda}{2}\,|\mathrm{tr}\,(\nabla u - p)|^2 + \underbrace{\frac{\mu\,k_1}{2}\,|p|^2 + \frac{\mu\,L_c^2}{2}\,|\mathrm{Curl}\,p|^2}_{\text{immediate } H(\mathrm{Curl}\,)\text{-control of } p}\,,$$

which is, however, not invariant w.r.t. the transformations

$$u \to u + \overline{A}x + \overline{b}\,, \qquad p \to p + \overline{A}\,,$$

for constant skew-symmetric $\overline{A} \in \mathfrak{so}(3)$ and constant translation $\overline{b} \in \mathbb{R}^3$, which represent superposed Euclidean motions on both the displacement and plastic distortion. Therefore, the choice $\mathrm{sym}\,p$ in the backstress evolution is mandatory by Euclidean invariance: the linear kinematic hardening must be based on a symmetric backstress tensor.

Before we present our analysis of the model with linear kinematical hardening and plastic spin, we find it important to first present, using the convex analytical setting, a summary of those few models of infinitesimal gradient plasticity in the literature for which a mathematical analysis is now available. Precisely, we present in Section 3 the model by Mühlhaus-Aifantis [33] as analyzed in [14, 15], the model by Gurtin-Anand [21] as studied in [46], the models with plastic spin analyzed in [34, 17] and their irrotational version in [41] and we highlight some interconnnections.

Let us first fix some notations and definitions which will make the paper more clear and readable.

2 Some notational agreements and definitions

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega$, which is occupied by an elastoplastic body in its undeformed configuration. Let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. A material point in Ω is denoted by x and the time domain under consideration is the interval $[0, T]$.
For every a, $b \in \mathbb{R}^3$, we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $|a|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{R}^{3\times3}$ the set of real 3×3 tensors. The standard Euclidean scalar product on $\mathbb{R}^{3\times3}$ is given by $\langle A, B\rangle_{\mathbb{R}^{3\times3}} = \mathrm{tr}\left[AB^T\right]$, where B^T denotes the transpose tensor of B. Thus, the Frobenius tensor norm is $|A|^2 = \langle A, A\rangle_{\mathbb{R}^{3\times3}}$. In the following we omit the subscripts $\mathbb{R}^3$ and $\mathbb{R}^{3\times3}$. The identity tensor on $\mathbb{R}^{3\times3}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}(A) = \langle A, \mathbb{1}\rangle$. We let $\mathrm{Sym}\,(3) := \{X \in \mathbb{R}^{3\times3} \mid X^T = X\}$ denote the set of symmetric tensors, the Lie-Algebras $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3\times3} \mid X^T = -X\}$ of skew-symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3\times3} \mid \mathrm{tr}\,(X) = 0\}$ of traceless tensors. For every $X \in \mathbb{R}^{3\times3}$, we set $\mathrm{sym}(X) = \frac{1}{2}\big(X + X^T\big)$, $\mathrm{skew}\,(X) = \frac{1}{2}\big(X - X^T\big)$ and $\mathrm{dev}(X) = X - \frac{1}{3}\mathrm{tr}\,(X)\,\mathbb{1} \in \mathfrak{sl}(3)$ for the symmetric part, the skew-symmetric part and the deviatoric part of X, respectively.

The body is assumed to undergo infinitesimal deformations. Its behaviour is governed by a set of equations and constitutive relations. Below is a list of variables and parameters involved in various models of infinitesimal gradient plasticity presented in this paper:

- u the displacement of the macroscopic material points;
- p the plastic distortion variable is a non-symmetric second order tensor, incapable of sustaining volumetric changes; that is, $p \in \mathfrak{sl}(3)$;
- $e = \nabla u - p$ the elastic distortion is a non-symmetric second order tensor;
- $\varepsilon^p = \operatorname{sym} p$ the symmetric plastic strain tensor;
- $\varepsilon^e = \operatorname{sym}(\nabla u - p)$ the symmetric elastic strain tensor;
- σ the Cauchy stress tensor is a symmetric second order tensor;
- σ_{y} the yield stress;
- f the body force;
- $\operatorname{Curl} p = -\operatorname{Curl} e$ the dislocation density tensor;
- τ^p the microstress tensor is a second order deviatoric symmetric tensor;
- $\mathfrak{m}^p = (m_{ijk})$ the micro-polar stress tensor is a third order tensor deviatoric symmetric in the first two indices i and j. That is, $m_{ijk} = m_{jik}$ and $m_{iik} = 0$;
- γ the accumulated plastic strain.

For isotropic media, the fourth order elasticity tensor $\mathbb{C}$ is given by

$$\mathbb{C}.X = 2\mu \operatorname{dev} \operatorname{sym} X + \kappa \operatorname{tr}(X)\mathbb{1} = 2\mu \operatorname{sym} X + \lambda \operatorname{tr}(X)\mathbb{1} \tag{2.1}$$

for any second-order tensor X, where μ and λ are the Lamé moduli satisfying

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0\,,$$

and $\kappa > 0$ is the bulk modulus.
These conditions suffice for pointwise ellipticity of the elasticity tensor in the sense that there exists a constant $m_0 > 0$ such that

$$\langle X, \mathbb{C}.X\rangle \geq m_0 |\operatorname{sym} X|^2\,. \tag{2.2}$$

The space of square integrable functions is $L^2(\Omega)$, while the Sobolev spaces used in this paper are:

$$\begin{aligned}
\mathrm{H}^1(\Omega) &= \{u \in L^2(\Omega)\,|\,\operatorname{grad} u \in L^2(\Omega)\}\,, && \operatorname{grad} = \nabla\,,\\
\|u\|^2_{H^1(\Omega)} &= \|u\|^2_{L^2(\Omega)} + \|\operatorname{grad} u\|^2_{L^2(\Omega)}\,, && \forall u \in \mathrm{H}^1(\Omega)\,,\\
\mathrm{H}(\operatorname{curl};\Omega) &= \{v \in L^2(\Omega)\,|\,\operatorname{curl} v \in L^2(\Omega)\}\,, && \operatorname{curl} = \nabla\times\,,\\
\|v\|^2_{\mathrm{H}(\operatorname{curl};\Omega)} &= \|v\|^2_{L^2(\Omega)} + \|\operatorname{curl} v\|^2_{L^2(\Omega)}\,, && \forall v \in \mathrm{H}(\operatorname{curl};\Omega)\,.
\end{aligned}$$

For every $X \in C^1(\Omega, \mathbb{R}^{3\times 3})$ with rows X^1, X^2, X^3, we use in this paper the definition of Curl X in [34, 47]:

$$\text{Curl } X = \begin{pmatrix} \text{curl}\, X_1 & - & - \\ \text{curl}\, X_2 & - & - \\ \text{curl}\, X_3 & - & - \end{pmatrix} \in \mathbb{R}^{3\times 3},$$

for which Curl $\nabla v = 0$ for every $v \in C^2(\Omega, \mathbb{R}^3)$. Notice that the definition of Curl X above is such that $(\text{Curl } X)^T a = \text{curl}\,(X^T a)$ for every $a \in \mathbb{R}^3$ and this clearly corresponds to the transpose of the Curl of a tensor as defined in [21, 22].

In Paragraph 3.5, we will need an explicit definition of the linear operator $\mathbb{L} : \mathbb{R}^{3\times 3\times 3} \to \mathbb{R}^{3\times 3}$ such that

$$\text{Curl } X = \mathbb{L}.\nabla X \qquad \forall X \in C^1(\Omega, \mathbb{R}^{3\times 3}). \tag{2.3}$$

So, for

$$\overline{A} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

we consider the operator $\text{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ through

$$\text{axl}(\overline{A}) := (a_1, a_2, a_3)^T, \qquad \overline{A}.\, v = (\text{axl}\, \overline{A}) \times v, \qquad \forall\, v \in \mathbb{R}^3,$$

$$(\text{axl}\, \overline{A})_k = -\frac{1}{2} \sum_{i,j=1}^{3} \epsilon_{ijk} \overline{A}_{ij} = \frac{1}{2} \sum_{i,j=1}^{3} \epsilon_{kij} \overline{A}_{ji},$$

where ϵ_{ijk} is the totally antisymmetric third order permutation tensor.
Hence, for every $A \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned}(\text{axl}\,\text{skew}\, A)_k = \frac{1}{2} \sum_{i,j=1}^{3} \epsilon_{kij}\, \text{skew}(A)_{ji} &= \frac{1}{4} \sum_{i,j=1}^{3} \epsilon_{kij} A_{ji} - \frac{1}{4} \sum_{i,j=1}^{3} \epsilon_{kij} A_j i \\ &= \frac{1}{2} \sum_{i,j=1}^{3} \epsilon_{kij} A_{ji}.\end{aligned}$$

Recalling that $(\text{curl } v)_k = \sum_{i,j=1}^{3} \epsilon_{kij} v_{j,i}$ for every $v \in C^1(\Omega, \mathbb{R}^3)$, it follows that

$$(\text{axl}\,\text{skew}\, \nabla v)_k = \frac{1}{2} \sum_{i,j=1}^{3} \epsilon_{kij} v_{j,i} = \frac{1}{2} (\text{curl } v)_k.$$

Therefore, we may rewrite

$$\text{Curl } X = \begin{pmatrix} 2\ \text{axl}\,\text{skew}\, \nabla X_1 & --- \\ 2\ \text{axl}\,\text{skew}\, \nabla X_2 & --- \\ 2\ \text{axl}\,\text{skew}\, \nabla X_3 & --- \end{pmatrix} = \mathbb{L}.\nabla X,$$

where $\mathbb{L} : \mathbb{R}^{3\times3\times3} \to \mathbb{R}^{3\times3}$ is given by

$$\mathbb{L} = (\widetilde{\mathbb{L}}_1, \widetilde{\mathbb{L}}_2, \widetilde{\mathbb{L}}_3)^T, \qquad \mathbb{L}.\nabla X = \begin{pmatrix} \widetilde{\mathbb{L}}_1.\nabla X \text{ ---} \\ \widetilde{\mathbb{L}}_2.\nabla X \text{ ---} \\ \widetilde{\mathbb{L}}_3.\nabla X \text{ ---} \end{pmatrix} \tag{2.4}$$

with $\widetilde{\mathbb{L}}_i : \mathbb{R}^{3\times3} \to \mathbb{R}^3$, $i = 1, 2, 3$ defined by

$$\widetilde{\mathbb{L}}_i.\nabla X = 2 \;\mathrm{axl}\,\mathrm{skew}\,\nabla X_i. \tag{2.5}$$

Hence, we have found the explicit linear operator $\mathbb{L} : \mathbb{R}^{3\times3\times3} \to \mathbb{R}^{3\times3}$, so that the equality (2.3) holds.
Notice, that for every $X,\, Y \in C^1(\Omega,\, \mathbb{R}^{3\times3})$

$$\begin{aligned} \langle \mathrm{Curl}\ X, \mathrm{Curl}\ Y\rangle &= \sum_{i=1}^{3} \langle \mathrm{curl}\ X_i, \mathrm{curl}\ Y_i\rangle = 4\sum_{i=1}^{3} \langle \mathrm{axl}\,\mathrm{skew}\,\nabla X_i, \mathrm{axl}\,\mathrm{skew}\,\nabla Y_i\rangle \\ &= 2\sum_{i=1}^{3} \langle \mathrm{skew}\,\nabla X_i, \mathrm{skew}\,\nabla Y_i\rangle = 2\sum_{i=1}^{3} \langle \mathrm{skew}\,\nabla X_i, \nabla Y_i\rangle\,. \end{aligned}$$

The following function spaces and norm will be used later.

$$\begin{aligned} &\mathrm{H}(\mathrm{Curl};\,\Omega,\,\mathbb{R}^{3\times3}) = \{X \in L^2(\Omega,\,\mathbb{R}^{3\times3}) \,|\, \mathrm{Curl}\,X \in L^2(\Omega,\,\mathbb{R}^{3\times3})\}\,, \\ &\|X\|^2_{\mathrm{H}(\mathrm{Curl};\Omega)} = \|X\|^2_{L^2(\Omega)} + \|\mathrm{Curl}\,X\|^2_{L^2(\Omega)}\,, \quad \forall X \in \mathrm{H}(\mathrm{Curl};\Omega,\,\mathbb{R}^{3\times3})\,, \\ &\mathrm{H}(\mathrm{Curl};\,\Omega,\,\mathbb{E}) := \{X : \Omega \to \mathbb{E} \,|\, X \in \mathrm{H}(\mathrm{Curl};\,\Omega,\,\mathbb{R}^{3\times3})\}\,, \end{aligned} \tag{2.6}$$

for $\mathbb{E} := \mathrm{Sym}\,(3)$, $\mathfrak{sl}(3)$ or $\mathrm{Sym}\,(3) \cap \mathfrak{sl}(3)$.

We also consider the space

$$\mathrm{H}_0(\mathrm{Curl};\,\Omega,\,\Gamma, \mathbb{R}^{3\times3})$$

as the completion in the norm in (2.6) of the space $\{q \in C^\infty(\Omega,\,\Gamma,\,\mathbb{R}^{3\times3}) \mid q \times \vec{n}|_\Gamma = 0\}$.
Therefore, this space generalizes the Dirichlet boundary condition

$$q \times \vec{n}|_\Gamma = 0$$

to be satisfied by the plastic distortion p or the plastic strain $\varepsilon^p := \mathrm{sym}\,p$. The space $\mathrm{H}_0(\mathrm{Curl};\,\Omega,\,\Gamma, \mathbb{E})$ is defined as in (2.6).

The divergence operator Div on second order tensor-valued functions is also defined row-wise as

$$\mathrm{Div}\,X = \begin{pmatrix} \mathrm{div}\,X_1 \\ \mathrm{div}\,X_2 \\ \mathrm{div}\,X_3 \end{pmatrix}\,.$$

3 Some models of infinitesimal gradient plasticity

3.1 The model by Mühlhaus-Aifantis [33]

In this model, the yield-stress in the case of isotropic hardening, is set to depend also on some derivative of a scalar measure of the accumulated plastic distortion which plays the role of the isotropic hardening variable. A summary of the model is presented in Table 2.

Additive split of strain:	$\operatorname{sym}\nabla u = \varepsilon = \varepsilon^e + \varepsilon^p\,, \quad \varepsilon^p \in \operatorname{Sym}(3)$
Equilibrium:	$\operatorname{Div}\sigma + f = 0$ with $\sigma = \mathbb{C}.\varepsilon^e$
Free energy:	$\frac{1}{2}\langle \mathbb{C}.\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\,\mu\, k_2 \lvert\gamma\rvert^2 + \frac{1}{2}\,\mu\, l^2 \lvert\nabla\gamma\rvert^2$
Yield condition:	$\phi(\sigma, g) = \lvert\operatorname{dev}\sigma\rvert + g - \sigma_{\mathrm{y}} \le 0$
where	$g = -\mu\, k_2\,\gamma + \mu\, l^2 \Delta\gamma$
Dissipation inequality:	$\int_\Omega \big[\langle \sigma, \dot\varepsilon^p\rangle + g\,\dot\gamma\big]\, dx \ge 0$
Dissipation function:	$\mathcal{D}(q,\xi) := \begin{cases} \sigma_{\mathrm{y}}\lvert q\rvert & \text{if } \lvert q\rvert \le \xi, \\ +\infty & \text{otherwise} \end{cases}$
Flow law in primal form:	$(\sigma, g) \in \partial\mathcal{D}(\dot\varepsilon^p, \dot\gamma)$.
Flow law in dual form:	$\dot\varepsilon^p = \lambda\, \dfrac{\operatorname{dev}\sigma}{\lvert\operatorname{dev}\sigma\rvert}, \quad \dot\gamma = \lambda = \lvert\dot\varepsilon^p\rvert$
KKT conditions:	$\lambda \ge 0, \quad \phi(\sigma, g) \le 0, \quad \lambda\,\phi(\sigma, g) = 0$
Boundary condition on γ:	$\gamma = 0$ on $\partial\Omega$
Function spaces for ε^p and γ:	$\varepsilon^p(t,\cdot) \in L^2(\Omega, \mathbb{R}^{3\times 3}), \quad \gamma(t,\cdot) \in H^1_0(\Omega)$

Table 2: The model by Mühlhaus-Aifantis [33] as formulated in [14, 15].

Under suitable boundary and initial conditions on u, ε^p and γ, the well-posedness as well as computational aspects of that model are studied in [14, 15] with the flow law formulated in its primal form.

3.2 The model by Gurtin and Anand [21] as studied in [46] with isotropic hardening

This model is based on the assumption that the power expended by each kinematical field be expressible in terms of a system of forces consistent with its own balance. Therefore, the model is characterized by two additional stress tensors: a second order tensor τ^p power conjugate to the symmetric plastic strain ε^p and a third order tensor $\mathfrak{m}^p$ power conjugate to the gradient of the plastic strain, which satisfy a microforce balance. The latter as well as the equilibrium being derived by the principle of virtual power. Since ε^p is deviatoric symmetric, it is not restrictive to assume that τ^p is deviatoric symmetric and the third order tensor $\mathfrak{m}^p$ is deviatoric symmetric in the first two indices. The model as formulated in [46] is summarized in Table 3 with the purely energetic version in Table 6.

Additive split of strain:	$\operatorname{sym}\nabla u = \varepsilon = \varepsilon^e + \varepsilon^p\,, \quad \varepsilon^p \in \operatorname{Sym}(3)$
Equilibrium:	$\operatorname{Div}\sigma + f = 0$ with $\sigma = \mathbb{C}.\varepsilon^e$
Microforce balance:	$\operatorname{dev}\sigma = \tau^p - \operatorname{Div}\mathfrak{m}^p,$
where	τ^p: microstress (2^{nd} order)
	$\mathfrak{m}^p$: micropolar stress (3^{rd} order)
Free energy:	$\frac{1}{2}\langle\mathbb{C}.\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\mu\, L_c^2\, \lvert\operatorname{Curl}\varepsilon^p\rvert^2 + \frac{1}{2}\mu\, k_2 \lvert\gamma\rvert^2$
Yield condition:	$\phi(\tau^p, \mathfrak{m}^p, g) := \sqrt{\lvert\tau^p\rvert^2 + \ell^{-2}\lvert\mathfrak{m}^p_{\text{diss}}\rvert^2} + g - \sigma_{\text{y}} \leq 0$
where	$\mathfrak{m}^p_{\text{diss}} = \mathfrak{m}^p - \mathfrak{m}^p_{\text{energ}}$
	$\mu L_c^2\, \langle\operatorname{Curl}\varepsilon^p, \operatorname{Curl}\dot{\varepsilon}^p\rangle = \langle\mathfrak{m}^p_{\text{energ}}, \nabla\dot{\varepsilon}^p\rangle$
Dissipation inequality:	$\int_\Omega \big[\langle\tau^p, \dot{\varepsilon}^p\rangle + \langle\mathfrak{m}^p_{\text{diss}}, \nabla\dot{\varepsilon}^p\rangle + g\dot{\gamma}\big]\,dx \geq 0, \quad g = -\mu\, k_2\, \gamma$
Dissipation function:	$\mathcal{D}(q,\xi) := \begin{cases} \sigma_{\text{y}}\, d^p(q) & \text{if } d^p(q) \leq \xi, \\ +\infty & \text{otherwise} \end{cases}$
where	$d^p(q) := \sqrt{\lvert q\rvert^2 + \ell^2\lvert\nabla q\rvert^2}$
Flow law in primal form:	$(\tau^p, \mathfrak{m}^p_{\text{diss}}, g) \in \partial\mathcal{D}(\dot{\varepsilon}^p, \nabla\dot{\varepsilon}^p, \dot{\gamma})$
Flow law in dual form:	$\left.\begin{array}{l} \dot{\varepsilon}^p = \lambda\,\dfrac{\tau^p}{\sigma_{\text{y}} - g}, \quad \nabla\dot{\varepsilon}^p = \lambda\,\ell^{-2}\,\dfrac{\mathfrak{m}^p_{\text{diss}}}{\sigma_{\text{y}} - g}, \\ \dot{\gamma} = \lambda = d^p(\dot{\varepsilon}^p) \end{array}\right\} \quad (*)$
KKT conditions:	$\lambda \geq 0, \quad \phi(\tau^p, \mathfrak{m}^p_{\text{diss}}, g) \leq 0, \quad \lambda\,\phi(\tau^p, \mathfrak{m}^p_{\text{diss}}, g) = 0$
Boundary conditions for ε^p:	$\varepsilon^p = 0$ on $\partial\Omega$
Function space for ε^p:	$\varepsilon^p(t,\cdot) \in H^1_0(\Omega, \operatorname{Sym}(3))$
Two length scales:	dissipative ℓ and energetic L_c

Table 3: The model by Gurtin and Anand [21] as formulated in [46] with dissipative **and** energetic length scales.

The well-posedness of the model was studied by Reddy et al. [46]. We would like to emphasize here that the starting point of the modelling and analysis is the primal form.

The corresponding solution will satisfy the dual form in which it is understood that there is an extra consistency condition generated which makes equation $(*)_2$ in Table 3 possible. The plastic strain variable ε^p is assumed from the outset to be symmetric. Note that the formulation in [21] as well as in [46] involves the full gradient $\nabla\varepsilon^p$ of the plastic strain in the dissipation function, which is controlled in L^2 leading then to find the plastic strain variable ε^p in the Sobolev space $H^1(\Omega, \mathrm{Sym}\,(3))$ together with the possibility to completely prescribe ε^p at the boundary.

3.3 The model with plastic spin in [17] and in [34]

Unlike the model in [21] with the microstresses and the plastic distortion kept symmetric, a model involving the plastic spin is studied in [34] with phenomenological Prager type kinematical hardening and in [17] with isotropic hardening. A summary of the setting in [17] for the so-called equal spin case is presented in Table 4.

Additive split of distortion:	$\nabla u = e + p, \quad \varepsilon^e = \mathrm{sym}\, e, \quad \varepsilon^p = \mathrm{sym}\, p$
Equilibrium:	$\mathrm{Div}\,\sigma + f = 0$ with $\sigma = \mathbb{C}.\varepsilon^e$
Free energy:	$\frac{1}{2}\langle \mathbb{C}.\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\mu\, L_c^2\, \lvert\mathrm{Curl}\, p\rvert^2 + \frac{1}{2}\mu\, k_2\, \lvert\gamma\rvert^2$
Yield condition:	$\phi(\Sigma_E, g) := \lvert\,\mathrm{dev}\,\Sigma_E\rvert + g - \sigma_{\mathrm{y}} \leq 0$
where	$\Sigma_E := \sigma + \Sigma_{\mathrm{curl}}^{\mathrm{lin}},\ \ \Sigma_{\mathrm{curl}}^{\mathrm{lin}} = -\mu\, L_c^2\ \mathrm{Curl\ Curl}\ p$
	$g = -\mu\, k_2\, \gamma$
Dissipation inequality:	$\displaystyle\int_\Omega [\langle\Sigma_E, \dot p\rangle + g\dot\gamma]\, dx \geq 0$
Dissipation function:	$\mathcal{D}(q,\xi) := \begin{cases} \sigma_{\mathrm{y}}\,\lvert q\rvert & \text{if } \lvert q\rvert \leq \xi, \\ \infty & \text{otherwise} \end{cases}$
Flow law in primal form:	$(\Sigma_E, g) \in \partial\mathcal{D}(\dot p, \dot\gamma)$
flow law in dual form:	$\dot p = \lambda\, \dfrac{\mathrm{dev}\,\Sigma_E}{\lvert\,\mathrm{dev}\,\Sigma_E\rvert}, \quad \dot\gamma = \lambda = \lvert\dot p\rvert$
KKT conditions:	$\lambda \geq 0, \quad \phi(\Sigma_E, g) \leq 0, \quad \lambda\,\phi(\Sigma_E, g) = 0$
Boundary conditions for p:	$p \times \vec n = 0$ on Γ, $(\mathrm{Curl}\ p) \times \vec n = 0$ on $\partial\Omega \setminus \Gamma$
Function space for p:	$p(t,\cdot) \in \mathrm{H}(\mathrm{Curl};\, \Omega,\, \mathbb{R}^{3\times 3})$

Table 4: The models with plastic spin in [17] and in [34].

An existence result for the weak formulation of this model is obtained in [17]. The solution space for this model is quite naturally $p \in H(\mathrm{Curl})$ since the isotropic hardening provides an L^2-control of the entire plastic distortion p and the energetic defect energy adds automatically a control of $\mathrm{Curl}\, p \in L^2$.

3.4 The irrotational version of [17].

In [41], the irrotational limit case has been computationally implemented as one of the first efficient treatments of gradient plasticity. In this model, the plastic distortion p remains symmetric and can therefore be written as $\varepsilon^p = \mathrm{sym}\, p$. A summary of the model is presented in Table 5. The well-posedness of this limit case is included in the analysis presented in [17].

As shown in the next paragraph, this model can also be obtained as a particular case of Gurtin-Anand [21] for $l = 0$, $L_c > 0$. Since the dissipative length scale $l = 0$, the solution space is only $H(\mathrm{Curl})$ with the attendant tangential boundary conditions. Thus, the existence result in [17] provides also the first existence result for the purely energetic Gurtin-Anand model with local isotropic hardening.

3.5 The Gurtin-Anand model: purely energetic version

In this section, we would like to compare or find a connection between the model by Gurtin-Anand and our irrotational version. To this aim, we consider the defect energy

$$\frac{1}{2}\mu\, L_c^2\, |\,\mathrm{Curl}\ \varepsilon^p|^2 = \frac{1}{2}\mu\, L_c^2\, |\mathbb{L}.\nabla\varepsilon^p|^2\,, \tag{3.1}$$

where the linear operator $\mathbb{L} : \mathbb{R}^{3\times3\times3} \to \mathbb{R}^{3\times3}$ explicitly defined in (2.4)-(2.5) is such that

$$\mathrm{Curl}\ \varepsilon^p = \mathbb{L}.\nabla\varepsilon^p\,.$$

We recall that

$$\mathrm{Curl}\ \varepsilon^p = \begin{pmatrix} \mathrm{curl}\ \varepsilon^p_{1---} \\ \mathrm{curl}\ \varepsilon^p_{2---} \\ \mathrm{curl}\ \varepsilon^p_{3---} \end{pmatrix}, \qquad \varepsilon^p_i, \quad i = 1,2,3 \text{ denote the rows of } \varepsilon^p \in \mathbb{R}^{3\times3}.$$

On the one hand, considering the variation $\delta\varepsilon^p_i \in C_0^\infty(\overline{\Omega}, \Gamma)$ of the left hand side of

Additive split of distortion:	$\nabla u = e + p, \quad \varepsilon^e := \operatorname{sym} e, \quad \varepsilon^p := \operatorname{sym} p$
Equilibrium:	$\operatorname{Div}\sigma + f = 0$ with $\sigma = \mathbb{C}.\varepsilon^e$
Free energy:	$\frac{1}{2}\langle \mathbb{C}.\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\mu\, L_c^2\, \lvert\operatorname{Curl}\varepsilon^p\rvert^2 + \frac{1}{2}\mu\, k_2\, \lvert\gamma\rvert^2$
Yield condition:	$\phi(\Sigma_E, g) := \lvert \operatorname{dev}\operatorname{sym}\Sigma_E\rvert + g - \sigma_{\mathrm{y}} \le 0$
where	$\Sigma_E := \sigma + \Sigma^{\mathrm{lin}}_{\mathrm{curl}}, \;\; \Sigma^{\mathrm{lin}}_{\mathrm{curl}} = -\mu\, L_c^2\, \operatorname{Curl}\operatorname{Curl}\varepsilon^p$
	$g = -\mu\, k_2\, \gamma$
Dissipation inequality:	$\int_\Omega [\langle \Sigma_E, \dot\varepsilon^p\rangle + g\dot\gamma]\, dx \ge 0$
Dissipation function:	$\mathcal{D}(q,\xi) := \begin{cases} \sigma_{\mathrm{y}}\lvert q\rvert & \text{if } \lvert q\rvert \le \xi, \\ \infty & \text{otherwise} \end{cases}$
Flow law in primal form:	$(\Sigma_E, g) \in \partial\mathcal{D}(\dot\varepsilon^p, \dot\gamma)$
flow law in dual form:	$\dot\varepsilon^p = \lambda \dfrac{\operatorname{dev}\operatorname{sym}\Sigma_E}{\lvert\operatorname{dev}\operatorname{sym}\Sigma_E\rvert}, \quad \dot\gamma = \lambda = \lvert\dot\varepsilon^p\rvert$
KKT conditions:	$\lambda \ge 0, \quad \phi(\Sigma_E, g) \le 0, \quad \lambda\, \phi(\Sigma_E, g) = 0$
Boundary conditions for ε^p:	$\varepsilon^p \times \vec{n} = 0$ on Γ, $(\operatorname{Curl}\varepsilon^p) \times \vec{n} = 0$ on $\partial\Omega \setminus \Gamma$
Function space for ε^p:	$\varepsilon^p(t,\cdot) \in \mathrm{H}(\mathrm{Curl};\, \Omega,\, \mathrm{Sym}(3))$

Table 5: The irrotational version of [17] with isotropic hardening. The plastic distortion itself does not appear, only $\varepsilon^p = \operatorname{sym} p$ remains in the model.

(3.1) with respect to the plastic strain variable we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\frac{1}{2}\int_\Omega \mu\, L_c^2 \lvert\operatorname{Curl}(\varepsilon^p + t\delta\varepsilon^p)\rvert^2 dx\Big|_{t=0} &= \mu\, L_c^2 \int_\Omega \langle \operatorname{Curl}\varepsilon^p, \operatorname{Curl}\delta\varepsilon^p\rangle\, dx \\
= \mu\, L_c^2 \sum_{i=1}^3 \int_\Omega \langle \operatorname{curl}\varepsilon^p_i, \operatorname{curl}\delta\varepsilon^p_i\rangle\, dx &= 4\,\mu\, L_c^2 \sum_{i=1}^3 \int_\Omega \langle \operatorname{axl}\operatorname{skew}\nabla\varepsilon^p_i, \operatorname{axl}\operatorname{skew}\nabla\delta\varepsilon^p_i\rangle\, dx \\
= 2\,\mu\, L_c^2 \sum_{i=1}^3 \int_\Omega \langle \operatorname{skew}\nabla\varepsilon^p_i, \operatorname{skew}\nabla\delta\varepsilon^p_i\rangle\, dx &= \sum_{i=1}^3 \int_\Omega \langle \underbrace{2\,\mu\, L_c^2 \operatorname{skew}\nabla\varepsilon^p_i}_{\mathfrak{m}^p_i,\ \text{2nd order tensor}}, \nabla\delta\varepsilon^p_i\rangle\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{i=1}^{3}\int_\Omega \langle \operatorname{Div}\mathfrak{m}_i^p, \delta\varepsilon_i^p\rangle\, dx + \sum_{i=1}^{3}\int_{\partial\Omega} \langle \mathfrak{m}_i^p.\vec{n}, \delta\varepsilon_i^p\rangle\, da\\
&= -\sum_{i=1}^{3}\int_\Omega \langle \operatorname{Div}\mathfrak{m}_i^p, \delta\varepsilon_i^p\rangle\, dx + \sum_{i=1}^{3}\int_{\partial\Omega} \langle (\operatorname{axl}\mathfrak{m}_i^p)\times\vec{n}, \delta\varepsilon_i^p\rangle\, da\\
&= -\sum_{i=1}^{3}\int_\Omega \langle \operatorname{Div}\mathfrak{m}_i^p, \delta\varepsilon_i^p\rangle\, dx - \sum_{i=1}^{3}\int_{\partial\Omega} \langle \operatorname{axl}\mathfrak{m}_i^p, (\delta\varepsilon_i^p)\times\vec{n}\rangle\, da\\
&= -\int_\Omega \langle \operatorname{Div}\mathfrak{m}^p, \delta\varepsilon^p\rangle\, dx - \sum_{i=1}^{3}\int_{\partial\Omega} \langle \operatorname{axl}\mathfrak{m}_i^p, (\delta\varepsilon_i^p)\times\vec{n}\rangle\, da, \qquad (3.2)
\end{aligned}
$$

where the third order tensor $\mathfrak{m}^p \in \mathbb{R}^{3\times3\times3}$ is defined by

$$
\mathfrak{m}^p = (\mathfrak{m}_1^p, \mathfrak{m}_2^p, \mathfrak{m}_3^p)^T. \qquad (3.3)
$$

On the other hand, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\frac{1}{2}\int_\Omega \mu\, L_c^2 |\operatorname{Curl}(\varepsilon^p + t\delta\varepsilon^p)|^2 dx\Big|_{t=0} = \mu\, L_c^2\int_\Omega \langle \operatorname{Curl}\,\varepsilon^p, \operatorname{Curl}\,\delta\varepsilon^p\rangle\, dx\\
&\qquad = \mu\, L_c^2\int_\Omega \langle \operatorname{Curl}\,\operatorname{Curl}\,\varepsilon^p, \delta\varepsilon^p\rangle\, dx + \mu\, L_c^2\sum_{i=1}^{3}\int_{\partial\Omega}\langle \delta\varepsilon_i^p \times [\operatorname{Curl}\,\varepsilon^p]_i, n\rangle\, da \qquad (3.4)\\
&\qquad = \mu\, L_c^2\int_\Omega \langle \operatorname{Curl}\,\operatorname{Curl}\,\varepsilon^p, \delta\varepsilon^p\rangle\, dx - \mu\, L_c^2\sum_{i=1}^{3}\int_{\partial\Omega}\langle [\operatorname{Curl}\,\varepsilon^p]_i, \delta\varepsilon_i^p \times n\rangle\, da.
\end{aligned}
$$

In view of (3.2) and (3.4), we obtain

$$
\mu\, L_c^2\int_\Omega \langle \operatorname{Curl}\,\operatorname{Curl}\,\varepsilon^p, \delta\varepsilon^p\rangle\, dx = -\int_\Omega \langle \operatorname{Div}\mathfrak{m}^p, \delta\varepsilon^p\rangle\, dx, \qquad (3.5)
$$

for all $\delta\varepsilon^p \in \mathrm{H}_0(\operatorname{Curl};\Omega;\Gamma)$, i.e., for $\delta\varepsilon_i^p \times \vec{n} = 0$.

Since we can assume that ε^p is trace free symmetric, so is $\delta\varepsilon^p$ and we may equivalently write

$$
\mu\, L_c^2\int_\Omega \langle \operatorname{dev}\,\operatorname{sym}\operatorname{Curl}\,\operatorname{Curl}\,\varepsilon^p, \delta\varepsilon^p\rangle\, dx = -\int_\Omega \langle \operatorname{Div}\mathfrak{m}^p, \delta\varepsilon^p\rangle\, dx\,.
$$

Thus, we get that

$$
-\operatorname{Div}\underbrace{\mathfrak{m}^p}_{\in\,\mathbb{R}^{3\times3\times3}} = \underbrace{\mu\, L_c^2\operatorname{dev}\,\operatorname{sym}\operatorname{Curl}\,\operatorname{Curl}\,\varepsilon^p}_{\in\,\mathbb{R}^{3\times3}}\,.
$$

Set

$$
\tau^p := \operatorname{dev}\sigma + \operatorname{Div}\mathfrak{m}^p = \underbrace{\operatorname{dev}\sigma - \mu\, L_c^2\operatorname{dev}\,\operatorname{sym}\operatorname{Curl}\,\operatorname{Curl}\,\varepsilon^p}_{=\operatorname{dev}\operatorname{sym}\Sigma_E}\,. \qquad (3.6)
$$

Hence, $\tau^p = \operatorname{dev}\operatorname{sym}\Sigma_E$.

Notice that the second order tensor $\operatorname{Div} m^p$ is trace free. In fact, from the bracket $\langle \mathfrak{m}^p, \nabla\varepsilon^p\rangle$, it holds that (we may assume) $m^p_{ijk} = m^p_{jik}$ (since ε^p is symmetric) and we nay also assume that

$$m^p_{iik} = 0\,. \tag{3.7}$$

Now, it is clear that $\operatorname{tr}(\operatorname{Div}\mathfrak{m}^p) = m_{iik,k} = 0$ from (3.7) and hence $\operatorname{Div}\mathfrak{m}^p$ is symmetric and trace free.

Find a summary of this model in Table 6.

Let us now repeat the formulation of the model with spin in more details, in its dual and primal setting for the paper to be rather self-contained.

4 The model with linear kinematical hardening and plastic spin

4.1 Strong formulation

The balance equation. The conventional macroscopic force balance leads to the equation of equilibrium

$$\operatorname{div}\sigma + f = \mathbf{0}\,. \tag{4.1}$$

Constitutive equations. The constitutive equations are obtained from a free energy imbalance together with a flow law that characterizes plastic behaviour. Since the model under study involves plastic spin, we consider an additive decomposition of the displacement gradient ∇u into elastic and plastic components e and p as mentioned in the notational section, so that

$$\nabla u = e + p \tag{4.2}$$

We consider here a free energy of the form

$$\begin{aligned}\Psi(\nabla u, p, \operatorname{Curl} p) :\;=\;& \underbrace{\Psi^{\text{lin}}_e(e)}_{\text{elastic energy}} + \underbrace{\Psi^{\text{lin}}_{\text{curl}}(\operatorname{Curl} p)}_{\text{defect energy (GND)}}\\ &+ \underbrace{\Psi^{\text{lin}}_{\text{kin}}(p)}_{\text{linear kinematical hardening energy}}\,,\end{aligned} \tag{4.3}$$

where

$$\Psi^{\text{lin}}_e(e) := \frac{1}{2}\langle\varepsilon^e, \mathbb{C}.\varepsilon^e\rangle, \quad \Psi^{\text{lin}}_{\text{curl}}(\operatorname{Curl} p) := \frac{1}{2}\mu\, L_c^2\,|\operatorname{Curl} p|^2 \text{ and}$$

$$\Psi^{\text{lin}}_{\text{kin}}(p) := \frac{1}{2}\mu\, k_1\,|\operatorname{dev}\operatorname{sym} p|^2\,.$$

Additive split of strain:	$\operatorname{sym}\nabla u = \varepsilon = \varepsilon^e + \varepsilon^p\,, \quad \varepsilon^p \in \operatorname{Sym}(3)$
Equilibrium:	$\operatorname{Div}\sigma + f = 0$ with $\sigma = \mathbb{C}.\varepsilon^e$
Microforce balance:	$\operatorname{dev}\sigma = \tau^p - \operatorname{Div}\mathfrak{m}^p,$
where	τ^p: microstress (2^{nd} order)
	$\mathfrak{m}^p$: micropolar stress (3^{rd} order)
	$\operatorname{tr}(\operatorname{Div}\mathfrak{m}^p) = 0$
	$\tau^p = \operatorname{dev}\sigma + \operatorname{Div}\mathfrak{m}^p \in \mathfrak{sl}(3) \cap \operatorname{Sym}(3)$
Free energy:	$\frac{1}{2}\langle \mathbb{C}.\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\mu\, L_c^2\, \lvert\operatorname{Curl}\varepsilon^p\rvert^2 + \frac{1}{2}\mu\, k_2 \lvert\gamma\rvert^2$
Yield condition:	$\phi(\tau^p, g) := \lvert\tau^p\rvert + g - \sigma_{\mathrm{y}} \le 0$
where	$\mathfrak{m}^p = \mathfrak{m}^p_{\text{energ}}$
	$\mu L_c^2\, \langle \operatorname{Curl}\varepsilon^p, \operatorname{Curl}\dot\varepsilon^p\rangle = \langle \mathfrak{m}^p, \nabla\dot\varepsilon^p\rangle$
Dissipation inequality:	$\int_\Omega \left[\langle \tau^p, \dot\varepsilon^p\rangle + g\dot\gamma\right] dx \ge 0, \quad g = -\mu\, k_2\, \gamma$
Dissipation function:	$\mathcal{D}(q,\xi) := \begin{cases} \sigma_{\mathrm{y}}\,\lvert q\rvert & \text{if } \lvert q\rvert \le \xi, \\ +\infty & \text{otherwise} \end{cases}$
Flow law in primal form:	$(\tau^p, g) \in \partial\mathcal{D}(\dot\varepsilon^p, \dot\gamma)$
Flow law in dual form:	$\dot\varepsilon^p = \lambda\, \dfrac{\tau^p}{\sigma_{\mathrm{y}} - g} = \dfrac{\tau^p}{\lvert\tau^p\rvert}, \qquad \dot\gamma = \lambda = \lvert\dot\varepsilon^p\rvert$
KKT conditions:	$\lambda \ge 0, \quad \phi(\tau^p, g) \le 0, \quad \lambda\, \phi(\tau^p, g) = 0$
Boundary conditions for ε^p:	$\varepsilon^p \times \vec{n} = 0$ on $\partial\Omega$
Function space for ε^p:	$\varepsilon^p(t,\cdot) \in \mathrm{H}(\operatorname{Curl}\,; \Omega, \operatorname{Sym}(3))$
Length scale:	energetic L_c

Table 6: The irrotational model by Gurtin and Anand [21] with no dissipative length scale i.e. $\mathfrak{m}^p_{\text{diss}} = 0$. Since τ^p has been identified with $\operatorname{dev}\operatorname{sym}\Sigma_E$ in (3.6), the model coincides with the irrotational version of [17].

L_c is the energetic length scale and k_1 is the dimensionless hardening modulus. The defect energy is related to geometrically necessary dislocations (GNDs) and the Burger's vector.

The local free-energy imbalance states that

$$\dot\Psi - \langle \sigma, \dot e\rangle - \langle \sigma, \dot p\rangle \le 0\,. \tag{4.4}$$

Now we expand the first term, substitute (4.3) and get

$$\langle \mathcal{C}\varepsilon^e - \sigma, \dot{\varepsilon}^e\rangle - \langle \sigma, \dot{p}\rangle + \mu\, L_c^2\, \langle \mathrm{Curl}\, p, \mathrm{Curl}\, \dot{p}\rangle + \mu\, k_1\, \langle \mathrm{dev\,sym}\, p, \dot{p}\rangle \leq 0\,, \tag{4.5}$$

which, using arguments from thermodynamics gives the elasticity relation

$$\sigma = \mathbb{C}.\varepsilon^e = 2\mu\ \mathrm{sym}(\nabla u - p) + \lambda\ \mathrm{tr}(\nabla - p)\mathbb{1} \tag{4.6}$$

and the reduced dissipation inequality

$$-\langle \sigma, \dot{p}\rangle + \mu\, L_c^2\, \langle \mathrm{Curl}\ p, \mathrm{Curl}\ \dot{p}\rangle + \mu\, k_1\, \langle \mathrm{dev\,sym}\, p, \dot{p}\rangle \leq 0. \tag{4.7}$$

Now we integrate (4.7) over Ω and get

$$\begin{aligned} 0 \quad &\geq \quad \int_\Omega \Big[-\langle \sigma, \dot{p}\rangle + \mu\, L_c^2\, \langle \mathrm{Curl}\ p, \mathrm{Curl}\ \dot{p}\rangle + \mu\, k_1\, \langle \mathrm{dev\,sym}\, p, \dot{p}\rangle \Big]\, dx \\ &= \quad \int_\Omega \Big[-\langle \sigma, \dot{p}\rangle + \mu\, L_c^2\, \langle \mathrm{Curl\ \ Curl}\ p, \dot{p}\rangle + \mu\, k_1\, \langle \mathrm{dev\,sym}\, p, \dot{p}\rangle \\ &\qquad + \sum_{i=1}^3 \mathrm{div}\Big(\mu\, L_c^2\, \frac{d}{dt} p^i \times (\mathrm{Curl}\ p)^i \Big) \Big]\, dx\,. \end{aligned} \tag{4.8}$$

Using the divergence theorem we obtain

$$\begin{aligned} &\int_\Omega \big[\langle -\sigma + \mu\, L_c^2\ \mathrm{Curl\ \ Curl}\ p, \dot{p}\rangle + \mu\, k_1\, \langle \mathrm{dev\,sym}\, p, \dot{p}\rangle \big]\ dx \\ &\qquad + \sum_{i=1}^3 \int_{\partial\Omega} \mu\, L_c^2\, \langle \dot{p}^i \times (\mathrm{Curl}\ p)^i, \vec{n}\rangle da \leq 0\,. \end{aligned} \tag{4.9}$$

In order to obtain a dissipation inequality in the spirit of classical plasticity, we assume that the infinitesimal plastic distortion p satisfies the so-called *linearized insulation condition*

$$\sum_{i=1}^3 \int_{\partial\Omega} \mu\, L_c^2\, \langle \frac{d}{dt} p^i \times (\mathrm{Curl}\ p)^i, \vec{n}\rangle da = 0\,. \tag{4.10}$$

This condition is satisfied if we assume for instance that the boundary is a perfect conductor. This means that the tangential component of p vanishes on $\partial\Omega$. In the context of dislocation dynamics these conditions express the requirement that there is no flux of the Burgers vector across a hard boundary. Gurtin and Anand [21] introduce the following different types of boundary conditions for the plastic distortion

$$\begin{aligned} \partial_t p \times \vec{n}|_{\Gamma_{\mathrm{hard}}} &= 0 \quad \text{"micro-hard" (perfect conductor)} \\ \partial_t p|_{\Gamma_{\mathrm{hard}}} &= 0 \quad \text{"hard-slip"} \\ \mathrm{Curl}\ p \times \vec{n}|_{\Gamma_{\mathrm{hard}}} &= 0 \quad \text{"micro-free"}\,. \end{aligned} \tag{4.11}$$

We specify a sufficient condition for the micro-hard boundary condition, namely

$$p \times \vec{n}|_{\Gamma_{\mathrm{hard}}} = 0 \tag{4.12}$$

and assume for simplicity only $\Gamma_{\rm hard} = \partial\Omega = \Gamma$. Note that this boundary condition constrains the plastic slip in tangential direction only, which is what we expect to happen at $\Gamma_{\rm hard}$.
Under (4.10), we then obtain the dissipation inequality

$$\int_\Omega \langle \sigma + \Sigma^{\rm lin}_{\rm curl} + \Sigma^{\rm lin}_{\rm kin}, \dot{p}\rangle \, dx \geq 0 \, , \tag{4.13}$$

where

$$\Sigma^{\rm lin}_{\rm curl} := -\mu \, L_c^2 \ {\rm Curl \ Curl} \, p \qquad \text{and} \qquad \Sigma^{\rm lin}_{\rm kin} := -\mu \, k_1 \, {\rm dev \, sym} \, p \, .$$

The flow law. We consider a yield function defined for every $\Sigma_{\rm E} := \sigma + \Sigma^{\rm lin}_{\rm curl} + \Sigma^{\rm lin}_{\rm kin}$ by

$$\phi_0(\Sigma_{\rm E}) := |\, {\rm dev} \, \Sigma_{\rm E}| - \sigma_{\rm y} \tag{4.14}$$

Here $\sigma_{\rm y}$ is the yield stress of the material. So the set of admissible (elastic) generalized stresses is

$$\mathcal{K}_0 := \left\{ \Sigma_{\rm E} = \sigma + \Sigma^{\rm lin}_{\rm curl} + \Sigma^{\rm lin}_{\rm kin} \mid \phi_0(\Sigma_{\rm E}) \leq 0 \right\} \, . \tag{4.15}$$

The maximum dissipation principle gives the normality law

$$\dot{p} \in N_{\mathcal{K}_0}(\Sigma_{\rm E})) \tag{4.16}$$

where $N_{\mathcal{K}_0}(\Sigma_{\rm E})$ denotes the normal cone to $\mathcal{K}_0$ at $\Sigma_{\rm E}$, which is the set of generalised strain rates $\dot{p}$ that satisfy

$$\langle \overline{\Sigma} - \Sigma_{\rm E}, \dot{p}\rangle \leq 0 \qquad \text{for all } \overline{\Sigma} \in \mathcal{K}_0 \, . \tag{4.17}$$

Notice that $N_{\mathcal{K}_0} = \partial \chi_0$ where χ_0 denotes the indicator function of the set $\mathcal{K}_0$ and $\partial\chi_0$ denotes the subdifferential of the function χ_0.
Whenever the yield surface $\partial\mathcal{K}_0$ is smooth at $\Sigma^{\rm p}$ then

$$\dot{p} \in N_{\mathcal{K}_0}(\Sigma_{\rm E})) \quad \Rightarrow \quad \exists \lambda \text{ such that } \dot{p} = \lambda \, \frac{{\rm dev} \, \Sigma_{\rm E}}{|\, {\rm dev} \, \Sigma_{\rm E}|}$$

with the Karush-Kuhn Tucker conditions: $\lambda \geq 0$, $\phi(\Sigma_{\rm E}) \leq 0$ and $\lambda \, \phi(\Sigma_{\rm E}) = 0$.

Using convex analysis (Legendre-transformation) we find that

$$\dot{p} \in \partial\chi_0(\Sigma_{\rm E}) \quad \Leftrightarrow \quad \Sigma_{\rm E} \in \partial\chi_0^*(\dot{p}) \, , \tag{4.18}$$

where χ_0^* is the Fenchel-Legendre dual of the function χ_0 denoted in this context by $\mathcal{D}$, the one-homogeneous dissipation function for rate-independent processes. That is,

$$\mathcal{D}(q) \ = \ \sup \left\{ \langle \sigma + \Sigma^{\rm lin}_{\rm curl} + \Sigma^{\rm lin}_{\rm kin}, q\rangle \ \mid \ \phi_0(\sigma + \Sigma^{\rm lin}_{\rm curl} + \Sigma^{\rm lin}_{\rm kin}) \leq 0 \right\} \ = \ \sigma_{\rm y} \, |q| \, . \tag{4.19}$$

We get from the definition of the subdifferential ($\Sigma_{\rm E} \in \partial\chi_0^*(\dot{p})$) that,

$$\mathcal{D}(q) \geq \mathcal{D}(\dot{p}) + \langle \Sigma_{\rm E}, q - \dot{p}\rangle \quad \text{for any } q. \tag{4.20}$$

That is,

$$\mathcal{D}(q) \geq \mathcal{D}(\dot{p}) + \langle \sigma + \Sigma^{\text{lin}}_{\text{curl}} + \Sigma^{\text{lin}}_{\text{kin}}, q - \dot{p}\rangle \ \text{ for any } q\,. \tag{4.21}$$

Strong formulation of the model. To summarize, we have obtained the following strong formulation for the model of infinitesimal gradient plasticity with kinematic hardening and plastic spin. The goal is to find:

(i) the displacement $u \in \mathsf{H}^1(0,T;\mathsf{H}^1_0(\Omega,\Gamma,\mathbb{R}^3))$,

(ii) the infinitesimal plastic distortion p with $\operatorname{sym} p \in \mathsf{H}^1(0,T;L^2(\Omega,\mathfrak{sl}(3)))$, $\operatorname{Curl} p \in \mathsf{H}^1(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))$ and $\operatorname{Curl}\operatorname{Curl} p \in \mathsf{H}^1(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))$

such that the content of Table 7 holds.

Additive split of distortion:	$\nabla u = e + p, \quad \varepsilon^e := \operatorname{sym} e,\ \varepsilon^p := \operatorname{sym} p$
Equilibrium:	$\operatorname{Div}\sigma + f = 0$ with $\sigma = \mathbb{C}.\varepsilon^e$
Free energy:	$\frac{1}{2}\langle \mathbb{C}.\varepsilon^e, \varepsilon^e\rangle + \frac{1}{2}\,\mu\, L_c^2\, \lvert\operatorname{Curl} p\rvert^2 + \frac{1}{2}\,\mu\, k_1\, \lvert \operatorname{dev}\operatorname{sym} p\rvert^2$
Yield condition:	$\phi(\Sigma_E) := \lvert \operatorname{dev}\Sigma_E\rvert - \sigma_{\text{y}} \leq 0$
where	$\Sigma_E := \sigma + \Sigma^{\text{lin}}_{\text{curl}} + \Sigma^{\text{lin}}_{\text{kin}}$,
	$\Sigma^{\text{lin}}_{\text{curl}} = -\mu\, L_c^2\ \operatorname{Curl}\operatorname{Curl} p,\ \Sigma^{\text{lin}}_{\text{kin}} = -\mu\, k_1\ \operatorname{dev}\operatorname{sym} p$
Dissipation inequality:	$\int_\Omega \langle \Sigma_E, \dot{p}\rangle dx \geq 0$
Dissipation function:	$\mathcal{D}(q) := \sigma_{\text{y}}\lvert q\rvert$
Flow law in primal form:	$\Sigma_E \in \partial\mathcal{D}(\dot{p})$
Flow law in dual form:	$\dot{p} = \lambda\, \dfrac{\operatorname{dev}\Sigma_E}{\lvert\operatorname{dev}\Sigma_E\rvert}, \qquad \lambda = \lvert\dot{p}\rvert$
KKT conditions:	$\lambda \geq 0, \quad \phi(\Sigma_E) \leq 0, \quad \lambda\,\phi(\Sigma_E) = 0$
Boundary conditions for p:	$p \times \vec{n} = 0$ on Γ, $(\operatorname{Curl} p) \times \vec{n} = 0$ on $\partial\Omega \setminus \Gamma$
Function space for p:	$p(t,\cdot) \in \mathrm{H}(\mathrm{Curl};\ \Omega,\ \mathbb{R}^{3\times3})$

Table 7: The model with linear kinematical hardening and plastic spin. The boundary condition on p necessitates at least $p \in \mathrm{H}(\mathrm{Curl};\ \Omega,\ \mathbb{R}^{3\times3})$. However, whether this is the case will only be proven at the end of the paper.

4.2 Weak formulation of the model

Assume that the strong formulation has a solution (u, p, γ). Let $v \in \mathsf{H}^1(\Omega, \mathbb{R}^3)$ with $v_{|\Gamma} = 0$. Multiply the equilibrium equation with $v - \dot{u}$ and integrate in space to get

$$\int_\Omega \langle \sigma, \nabla v - \nabla \dot{u} \rangle dx = \int_\Omega f(v - \dot{u}) dx\,. \tag{4.22}$$

Using the symmetry of the stress tensor σ and the elasticity relation we get

$$\int_\Omega \langle \mathbb{C}.\operatorname{sym}(\nabla u - p), \operatorname{sym}(\nabla v - \nabla \dot{u}) \rangle dx = \int_\Omega f(v - \dot{u}) dx\,. \tag{4.23}$$

Now, we take any $q \in C^\infty(\overline{\Omega}, \mathfrak{sl}(3))$ such that $q \times \vec{n} = 0$ on Γ and we integrate (4.21) over Ω, integrate by parts the term with Curl Curl using the boundary conditions

$$(q - \dot{p}) \times \vec{n} = 0 \text{ on } \Gamma$$

and get

$$\begin{aligned}
\int_\Omega \mathcal{D}_0^{\text{kin}}(q)\, dx \;\geq\; & \int_\Omega \mathcal{D}_0^{\text{kin}}(\dot{p})\, dx + \int_\Omega \langle \sigma + \Sigma_{\text{curl}}^{\text{lin}} + \Sigma_{\text{kin}}^{\text{lin}}, q - \dot{p} \rangle\, dx \\
\geq\; & \int_\Omega \mathcal{D}_0^{\text{kin}}(\dot{p})\, dx + \int_\Omega \langle \mathbb{C}.\operatorname{sym}(\nabla u - p), \operatorname{sym}(q - \dot{p}) \rangle\, dx \\
& - \int_\Omega \langle \mu L_c^2 \operatorname{Curl}\ \operatorname{Curl} p + \mu k_1 \operatorname{dev}\operatorname{sym} p, q - \dot{p} \rangle\, dx \qquad (4.24) \\
\geq\; & \int_\Omega \mathcal{D}_0^{\text{kin}}(\dot{p})\, dx + \int_\Omega \langle \mathbb{C}.\operatorname{sym}(\nabla u - p), \operatorname{sym}(q - \dot{p}) \rangle\, dx \\
& - \mu L_c^2 \int_\Omega \langle \operatorname{Curl} p, \operatorname{Curl}(q - \dot{p}) \rangle\, dx - \mu k_1 \int_\Omega \langle \operatorname{sym} p, q - \dot{p} \rangle\, dx\,.
\end{aligned}$$

Adding (4.24) to the weak formulation of the equilibrium in (4.23), we get that

$$\begin{aligned}
& \int_\Omega \Big[\langle \mathbb{C}.\operatorname{sym}(\nabla u - p), \operatorname{sym}(\nabla v - q) - \operatorname{sym}(\nabla \dot{u} - \dot{p}) \rangle + \mu L_c^2 \langle \operatorname{Curl} p, \operatorname{Curl}(q - \dot{p}) \rangle \\
& \quad + \mu k_1 \langle \operatorname{sym} p, \operatorname{sym} q - \operatorname{sym} \dot{p} \rangle \Big] dx + \int_\Omega \mathcal{D}_0^{\text{kin}}(q)\, dx - \int_\Omega \mathcal{D}_0^{\text{kin}}(\dot{p})\, dx \\
& \quad \geq \int_\Omega f(v - \dot{u})\, dx \qquad \forall (v, q)\,.
\end{aligned} \tag{4.25}$$

4.3 Existence result for the new formulation

To prove the existence result for the weak formulation (4.25), we follow the abstract machinery developed by Han and Reddy in [24] for mathematical problems in classical plasticity and used for instance in Djoko et al. [14], Reddy et al. [46], Neff et al. [34], Ebobisse-Neff [17] for models of gradient plasticity. To this aim, (4.25) is written as the variational inequality of the second kind: find $w = (u,p) \in \mathsf{H}^1(0,T;Z)$ such that $w(0) = 0$ and

$$\boldsymbol{a}(\dot{w}, z - w) + j_0(z) - j_0(\dot{w}) \geq \langle \ell, z - \dot{w} \rangle \text{ for every } z \in Z \text{ and for a.e. } t \in [0,T]\,, \tag{4.26}$$

where Z is a suitable Hilbert space to be constructed later,

$$\boldsymbol{a}(w,z) := \int_\Omega \Big[\langle \mathbb{C}.(\mathrm{sym}(\nabla u - p)), \mathrm{sym}(\nabla v - q) \rangle + \mu\, L^2 \langle \mathrm{Curl}\; p, \mathrm{Curl}\; q \rangle \tag{4.27}$$
$$+ \mu\, k_1 \langle \mathrm{sym}\, p, \mathrm{sym}\, q \rangle \Big]\, dx\,,$$

$$j_0(z) := \int_\Omega \mathcal{D}_0^{\mathrm{kin}}(q)\, dx\,, \tag{4.28}$$

$$\langle \ell, z \rangle := \int_\Omega f v\, dx\,, \tag{4.29}$$

for $w = (u,p)$ and $z = (v,q)$ in Z.
The Hilbert space Z is constructed in such a way that the functionals $\boldsymbol{a}$, j_0 and ℓ satisfy the assumptions in the abstract result in [24, Theorem 7.3]. The key issue here is the coercivity of the bilinear form $\boldsymbol{a}$ on Z. From the structure of the bilinear form $\boldsymbol{a}$ and the functional j_0, a natural attempt for the space of infinitesimal plastic distortions, is to consider the closure $\mathsf{H}_{\mathrm{sym}}\,(\mathrm{Curl}\,, \Omega, \Gamma; \mathfrak{sl}(3))$ of the linear subspace

$$\{q \in C^\infty(\overline{\Omega}, \mathbb{R}^{3\times3}) \mid \ \mathrm{tr}\, q = 0,\ q \times \vec{n} = 0 \text{ on } \Gamma\}$$

with respect to the norm

$$\|q\|^2_{\mathrm{sym,\,curl}} := \|\mathrm{sym}\, q\|^2_{L^2} + \|\mathrm{Curl}\, q\|^2_{L^2}\ . \tag{4.30}$$

Motivated by the well-posedness question for our model [34, 17], Neff et al. [37, 38, 39, 40], derived a new inequality extending Korn's inequality to incompatible tensor fields, namely there exist a constant $C(\Omega) > 0$ such that

$$\forall\, p \in \mathsf{H}(\mathrm{Curl}\,;\, \Omega,\, \mathbb{R}^{3\times3}) \mid \quad p \times \vec{n}|_\Gamma = 0: \tag{4.31}$$
$$\underbrace{\|p\|_{L^2(\Omega)}}_{\text{plastic distortion}} \leq C(\Omega) \left(\underbrace{\|\,\mathrm{sym}\, p\|_{L^2(\Omega)}}_{\text{plastic strain}} + \underbrace{\|\,\mathrm{Curl}\; p\|_{L^2(\Omega)}}_{\text{dislocation density}} \right).$$

Here, $\Gamma \subset \partial\Omega$ with full two-dimensional surface measure and the domain Ω needs to be **sliceable**, i.e. cuttable into finitely many simply connected subdomains with Lipschitz boundaries. The inequality (4.31) expresses the important fact that controlling the plastic strain $\mathrm{sym}\, p$ and the dislocation density $\mathrm{Curl}\; p$ in $L^2(\Omega)$ gives a control of the full plastic distortion p in $L^2(\Omega)$ provided the correct boundary conditions are specified: namely the micro-hard boundary condition. Since in the sequel we assume that $\mathrm{tr}(p) = 0$ (plastic incompressibility) the quadratic terms in the thermodynamic potential provide a control of the right hand side in (4.31). So, setting:

$$\begin{aligned} \mathsf{V} &= \mathsf{H}^1_0(\Omega,\Gamma,\mathbb{R}^3) = \{v \in \mathsf{H}^1(\Omega,\mathbb{R}^3) \,|\, v_{|\Gamma} = 0\}, && (4.32)\\ \mathsf{Q} &= \mathrm{H}_0(\mathrm{Curl};\, \Omega,\, \Gamma, \mathfrak{sl}(3)) = \mathsf{H}_{\mathrm{sym}}\,(\mathrm{Curl}\,;\, \Omega, \Gamma; \mathfrak{sl}(3)), && (4.33)\\ \mathsf{Z} &= \mathsf{V} \times \mathsf{Q}\,, && (4.34)\end{aligned}$$

equipped with the norms

$$\begin{aligned} &\|v\|_V := \|\nabla v\|_{L^2}\,, \qquad \|q\|^2_Q := \|\mathrm{sym}\, q\|^2_{L^2} + \|\mathrm{Curl}\, q\|^2_{L^2}\,, && (4.35)\\ &\|z\|^2_Z := \|v\|^2_V + \|q\|^2_Q \qquad \text{for } z = (v,q) \in \mathsf{Z}\,. && (4.36)\end{aligned}$$

Let us show that the bilinear form $\boldsymbol{a}$ is coercive on Z. Let therefore $z = (v, q) \in \mathsf{Z}$.

$$\begin{aligned} \boldsymbol{a}(z,z) &\geq m_0\, \|\mathrm{sym}\ \nabla v - \mathrm{sym}\, q\|^2_2 + \mu\, L_c^2\, \|\mathrm{Curl}\ q\|^2_2 + \mu\, k_1\, \|\mathrm{sym}\, q\|^2_2 \ \text{ (from (2.2))}\\ &= m_0 \Big[\|\mathrm{sym}\ \nabla v\|^2_2 + \|\mathrm{sym}\, q\|^2_2 - 2\langle \mathrm{sym}\ \nabla v, \mathrm{sym}\, p\rangle\Big]\\ &\qquad + \mu L_c^2\ \|\mathrm{Curl}\ q\|^2_2 + \mu\, k_1\, \|\mathrm{sym}\, q\|^2_2\\ &\geq m_0 \left[\|\mathrm{sym}\ \nabla v\|^2_2 + \|\mathrm{sym}\, q\|^2_2 - \theta\, \|\mathrm{sym}\ \nabla v\|^2_2 - \frac{1}{\theta}\, \|\mathrm{sym}\, q\|^2_2\right]\\ &\qquad + \mu\, L_c^2\, \|\mathrm{Curl}\ q\|^2_2 + \mu\, k_1\, \|\mathrm{sym}\, q\|^2_2 \qquad \text{(using Young's inequality)}\\ &= m_0(1-\theta)\, \|\mathrm{sym}\ \nabla v\|^2_2 + \left[m_0\Big(1 - \frac{1}{\theta}\Big) + \mu\, k_1\right] \|\mathrm{sym}\, q\|^2_2 + \mu\, L_c^2\, \|\mathrm{Curl}\ q\|^2_2\,. \end{aligned}$$

So, choosing θ such that $\dfrac{m_0}{m_0 + \mu k_1} < \theta < 1$ and using Korn's first inequality, we find a positive constant $C(m_0, \mu, k_1, L_c, \Omega) > 0$ such that

$$\boldsymbol{a}(z,z) \geq C\left[\|v\|^2_V + \|\mathrm{sym}\, q\|^2_2 + \|\mathrm{Curl}\ q\|^2_2\right] = C\, \|z\|^2_Z \quad \forall z = (v,q) \in \mathsf{Z}\,,$$

which proves the coercivity of our bilinear form and the inequality (4.31) shows the equivalence $\mathsf{Q} = \mathsf{H}_{\mathrm{sym}}(\mathrm{Curl}\,, \Omega, \Gamma; \mathfrak{sl}(3))$.

5 The Gurtin-Anand model with linear kinematical hardening: purely energetic version

Constitutive equations.

$$\nabla u = e + p \quad \Rightarrow \quad \mathrm{sym}\,\nabla u = \varepsilon^e + \varepsilon^p\,. \tag{5.1}$$

We consider here a free energy of the form

$$\begin{aligned}\Psi(\varepsilon^e, \varepsilon^p, \mathrm{Curl}\ \varepsilon^p) : &= \underbrace{\Psi^{\mathrm{lin}}_e(\varepsilon^e)}_{\text{elastic energy}} + \underbrace{\Psi^{\mathrm{lin}}_{\mathrm{curl}}(\mathrm{Curl}\ \varepsilon^p)}_{\text{defect energy (GND)}} \\ &\quad + \underbrace{\Psi^{\mathrm{lin}}_{\mathrm{kin}}(\varepsilon^p)}_{\text{linear kinematical hardening energy}}\,,\end{aligned} \tag{5.2}$$

where

$$\Psi_e^{\rm lin}(\varepsilon^e) := \frac{1}{2}\langle \varepsilon^e, \mathbb{C}.\varepsilon^e\rangle, \quad \Psi_{\rm curl}^{\rm lin}({\rm Curl}\,\varepsilon^p) := \frac{1}{2}\mu\, L_c^2\,|\,{\rm Curl}\,\varepsilon^p|^2 \quad \text{and}$$

$$\Psi_{\rm kin}^{\rm lin}(\varepsilon^p) := \frac{1}{2}\mu\, k_1\,|\,{\rm dev}\,\varepsilon^p|^2 .$$

Following the development in section 4, the free energy imbalance taking into account the boundary condition of the plastic strain variable

$$\varepsilon^p \times \vec{n}|_\Gamma = 0$$

leads to the dissipation inequality

$$\int_\Omega \langle \Sigma_E, \dot\varepsilon^p\rangle\, dx \geq 0\,, \tag{5.3}$$

where

$$\Sigma_E := \sigma - \mu\, k_1\ {\rm dev}\,\varepsilon^p - \mu\, L_c^2\,{\rm Curl}\ {\rm Curl}\ \varepsilon^p\,.$$

The flow law. The set of generalized stresses is

$$\mathcal{K} := \{\Sigma \in {\rm Sym}\,(3)\ |\ \ \phi(\Sigma) := |\,{\rm dev}\,\Sigma| - \sigma_{\rm y} \leq 0\}\,.$$

Hence, following [17], we get the flow in dual form

$$\dot\varepsilon^p \in N_{\mathcal{K}}(\Sigma_E) \tag{5.4}$$

where $N_{\mathcal{K}}(\Sigma_E)$ denotes the normal cone to $\mathcal{K}$ at Σ_E, which in case of smoothness reads as

$$\dot\varepsilon^p = \lambda\,\frac{{\rm dev}\ \Sigma_E}{|\,{\rm dev}\ \Sigma_E|}$$

with $\lambda \geq 0$, $\phi(\Sigma_E) \leq 0$, $\lambda\,\phi(\Sigma_E) = 0$.
The flow law in its primal formulation reads as

$$\Sigma_E \in \partial\mathcal{D}(\dot\varepsilon^p)\,.$$

That is,

$$\mathcal{D}(q)\ \geq\ \mathcal{D}(\dot\varepsilon^p) + \langle \Sigma_E, q - \dot\varepsilon^p\rangle \quad \forall\, q \in {\rm Sym}\,(3)\,, \tag{5.5}$$

where $\mathcal{D}$ is the dissipation function defined as

$$\mathcal{D}(q) := \{\langle \Sigma, q\rangle\ |\ \Sigma \in \mathcal{K}\} = \sigma_{\rm y}\,|q| \quad \forall\, q \in {\rm Sym}\,(3)\,.$$

Weak formulation of the model. Now arguing as in Section 4 and also as in Ebobisse-Neff [17, Section 3], we obtain a weak formulation of the model in the form of a variational inequality

$$\int_\Omega \Big[\langle \mathbb{C}.({\rm sym}\,\nabla u - \varepsilon^p), ({\rm sym}\,\nabla v - q) - ({\rm sym}\,\nabla \dot u - \dot\varepsilon^p)\rangle + \mu\, L_c^2\langle {\rm Curl}\ \varepsilon^p, {\rm Curl}\,(q - \dot\varepsilon^p)\rangle$$
$$+\mu\, k_1\langle \varepsilon^p, q - \dot\varepsilon^p\rangle\Big]\, dx + \int_\Omega \mathcal{D}(q)\, dx - \int_\Omega \mathcal{D}(\dot\varepsilon^p)\, dx$$
$$\geq \int_\Omega f(v - \dot u)\, dx \qquad \forall (v, q)\,. \tag{5.6}$$

The existence and uniqueness result for the variational inequality is easily obtained in the spaces

$$u \in \mathrm{H}^1(0,T;\, \mathrm{H}^1_0(\Omega,\, \Gamma,\, \mathbb{R}^3))\,,$$
$$\varepsilon^p \in \mathrm{H}^1(0,T;\, \mathrm{H}_0(\mathrm{Curl}\,,\Gamma;\mathrm{Sym}(3)\cap\mathfrak{sl}(3)))\,,$$

as in Section 4 through [24, Theorem 7.3], following the coercivity on the space

$$Z := \mathrm{H}^1_0(\Omega,\, \Gamma,\, \mathbb{R}^3)\times \mathrm{H}_0(\mathrm{Curl}\,,\Gamma;\mathrm{Sym}(3)\cap\mathfrak{sl}(3))\,,$$

of the bilinear form

$$\mathbf{a}(w,z) := \int_\Omega \Big[\langle \mathbb{C}.(\mathrm{sym}\,\nabla u - p), (\mathrm{sym}\,\nabla v - q)\rangle + \mu\, L_c^2 \langle \mathrm{Curl}\; p, \mathrm{Curl}\; q\rangle + \mu\, k_1 \langle p, q\rangle\Big]\, dx$$

for every $w = (u,p)$, $z = (v,q)$ in Z.
Note that since ε^p is already trace-free and symmetric, the coercivity of the bilinear form $\mathbf{a}$ does not the new Korn's type inequality in [37, 38, 39, 40], and in (4.31).

6 The infinitesimal elastic micromorphic model

The same total energy

$$\begin{aligned}\mathcal{E}(u,p) \;\;=\;\; \int_\Omega \Big[&\langle \mathbb{C}.\,\mathrm{sym}(\nabla u - p), \mathrm{sym}(\nabla u - p)\rangle \\ &+\frac{\mu\, k_1}{2}|\,\mathrm{dev}\,\mathrm{sym}\, p|^2 + \frac{\mu\, L_c^2}{2}|\,\mathrm{Curl}\; p|^2 - \langle f, u\rangle\Big]\, dx \qquad (6.1)\end{aligned}$$

is the starting point for a two-field minimization formulation

$$\mathcal{E}(u,p) \quad\to\quad \text{min. w.r.t } (u,p)\,,$$

in the sense of a micromorphic model ([35, 36]).
The relation of (6.1) to our plasticity formulation (1.2) is that in (6.1) the micromorphic distortion p is determined directly by a global energy minimization instead of a plastic flow rule. The microbalance equation is obtained as follows. The first variation of (6.1) with respect to p gives

$$\begin{aligned}&\int_\Omega \big[\langle \mathbb{C}.\,\mathrm{sym}(\nabla u - p), \mathrm{sym}\;\delta p\rangle + \mu\, k_1 \langle \mathrm{dev}\,\mathrm{sym}\; p, \delta p\rangle + \mu\, L_c^2\, \langle \mathrm{Curl}\;\; \mathrm{Curl}\;\; p, \delta p\rangle\big]\, dx \\ &= \int_\Omega \big[\langle \mathrm{sym}\;\mathbb{C}.\,\mathrm{sym}(\nabla u - p), \delta p\rangle + \langle \mu\, k_1, \mathrm{dev}\,\mathrm{sym}\; p, \delta p\rangle \\ &\qquad + \langle \mu\, L_c^2\;\mathrm{Curl}\;\;\mathrm{Curl}\;\; p, \delta p\rangle\big]\, dx = 0\,.\end{aligned}$$

The "microbalance" is then of the form

$$\mu\, L_c^2\;\mathrm{Curl}\;\;\mathrm{Curl}\;\; p = \overbrace{\mathrm{sym}\;\mathbb{C}.\,\mathrm{sym}(\nabla u - p)}^{\text{Cauchy stress }\sigma} - \mu\, k_1\;\, \mathrm{dev}\,\mathrm{sym}\; p\,.$$

The well-posedness of such a model has been shown in Neff et al. in [36]. Hence, we get in this model

$$\mu\, L_c^2 \text{ Curl Curl } p = \sigma - \mu\, k_1 \operatorname{dev}\operatorname{sym} p$$

or

$$0 = \underbrace{\sigma - \mu\, k_1 \operatorname{dev}\operatorname{sym} p - \mu\, L_c^2 \text{ Curl Curl } p}_{=\Sigma_E}$$

with

$$p \times \vec{n}|_\Gamma = 0 \quad \text{ and } \quad (\text{Curl } p) \times \vec{n}|_{\partial\Omega\setminus\Gamma} = 0\,,$$

instead of a dual flow law $\dot{p} \in \partial\mathcal{X}(\Sigma_E)$ in plasticity.

7 Conclusion

The development of the model with plastic spin is straightforward and involves only the addition of a quadratic defect energy. The boundary conditions on the plastic distortion are consistent both from the physical and the mathematical point of view. The departure from classical plasticity is minimal. Choosing a symmetric local kinematical backstress evolution necessitates to use a new Korn's type inequality for incompatible plastic distortions. Contrary to the presented alternative models, in which the energetic length scale L_c has only a "passive" role in that necessary estimates are already obtained from the dissipative length scale ℓ, in this model it is only the interplay between the energetic length scale and the symmetric local backstress which makes the problem well-posed. By identifying the irrotational Gurtin-Anand model with only energetic length scale as a special limit of our model with spin, we have been able to provide an existence theorem for that model for both the isotropic hardening case ([17]), as well as the local backstress case (this paper with the same considerations as in [17]). Moreover, our derivation of the model avoids the introduction of certain additional "micro force balances". Let us also mention that, the introduction of the irrotationality constraint appears, in our general framework with spin to be neither advantageous nor necessary, but simplifies the analysis considerably.

It remains to be seen if, in the dual formulation of our model with spin one may consider isotropic hardening driven by a symmetrized measure of accumulated plastic straining

$$\dot{\gamma} = |\operatorname{sym}\dot{p}|\,. \tag{7.1}$$

This would be conceptionally pleasing since kinematical hardening could then exclusively be related to the GND-distribution via the energetic length scale L_c (and assuming $k_1 = 0$) while the SSD-distribution would be described by the accumulated plastic straining. We need to remark that (7.1) does not seem to satisfy the additional assumption of *maximal dissipation*, making it unsuitable to be considered in the primal formulation. However, it is well-established that the equivalence of the primal and dual formulation is not satisfied in general for gradient plasticity.

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