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Global existence in rate-independent infinitesimal strain gradient plasticity

Patrizio Neff[1 *]

[1] Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstraße 7, 64289 Darmstadt, Germany.

We propose a model of infinitesimal strain gradient plasticity including phenomenological Prager type linear kinematical hardening and nonlocal kinematical hardening due to dislocation interaction. The model is a thermodynamically admissible model of infinitesimal plasticity involving only the Curl of the non-symmetric plastic distortion p. Linearized spatial and material covariance under constant infinitesimal rotations is satisfied. Uniqueness of strong solutions of the infinitesimal model is obtained if two non-classical boundary conditions on the plastic distortion p are introduced: $\dot{p}.\tau = 0$ on the microscopically hard boundary $\Gamma_D \subset \partial\Omega$ and $[\mathrm{Curl}\, p].\tau = 0$ on the microscopically free boundary $\partial\Omega \setminus \Gamma_D$, where τ are the tangential vectors at the boundary $\partial\Omega$. Moreover, a weak reformulation of the infinitesimal model allows for a global in-time solution of the corresponding rate-independent initial boundary value problem. The method of choice are a formulation as a quasi-variational inequality with symmetric and coercive bilinear form. Use is made of new Hilbert-space suitable for dislocation density dependent plasticity.

1 Introduction

The proposed infinitesimal strain gradient plasticity model is derived from a finite-strain strain gradient model based on the well-known multiplicative decomposition of the deformation gradient F into elastic and plastic parts. The underlying finite strain model involves a thermodynamically admissible flow rule for F_p which incorporates as plastic gradient $\mathrm{Curl}\, F_p$. This formulation is covariant w.r.t. superposed rigid rotations of the reference, intermediate and spatial configuration but the model is not spin-free due to the nonlocal dislocation interaction and cannot be reduced to a dependence on the plastic metric $C_p = F_p^T F_p$. The linearization leads to a thermodynamically admissible model of infinitesimal plasticity involving only the Curl of the non-symmetric plastic distortion p, see [1]

The corresponding linearized model can be obtained by writing down the corresponding quadratic potential in linearized quantities. Thus we expand $F = 1\!\!1 + \nabla u,\ F_p = 1\!\!1 + p + \ldots,\ F_e = 1\!\!1 + e + \ldots$ and the multiplicative decomposition turns into

$$\begin{aligned} 1\!\!1 + \nabla u &= (1\!\!1 + e + \ldots)(1\!\!1 + p + \ldots) \ \rightsquigarrow\ \nabla u \approx e + p + \ldots\,, \\ F_e^T F_e - 1\!\!1 &= 1\!\!1 + 2\,\mathrm{sym}\, e + e^T e - 1\!\!1 \ \rightsquigarrow\ 2\,\mathrm{sym}\, e = 2\,\mathrm{sym}(\nabla u - p)\,. \end{aligned} \tag{1}$$

Hence one obtains to highest order the **additive decomposition** of the displacement gradient $\nabla u = e + p$, with $\mathrm{sym}\, e = \mathrm{sym}(\nabla u - p)$ the **infinitesimal elastic lattice strain**, $\mathrm{skew}\, e = \mathrm{skew}(\nabla u - p)$ the **infinitesimal elastic lattice rotation** and $\kappa_e = \nabla\, \mathrm{axl}(\mathrm{skew}\, e)$ the **infinitesimal elastic lattice curvature** and p the **infinitesimal plastic distortion**. The quadratic energy which we use is given by

$$\begin{aligned} W(\nabla u, p, \mathrm{Curl}\, p) &= W_{\mathrm{e}}^{\mathrm{lin}}(\nabla u - p) + W_{\mathrm{ph}}(p) + W_{\mathrm{curl}}^{\mathrm{lin}}(\mathrm{Curl}\, p)\,, \\ W_{\mathrm{e}}^{\mathrm{lin}}(\nabla u - p) &= \mu \|\, \mathrm{sym}(\nabla u - p)\|^2 + \frac{\lambda}{2} \mathrm{tr}\, [\nabla u - p]^2\,, \\ W_{\mathrm{ph}}^{\mathrm{lin}}(p) = \mu\, h^+ \|\, \mathrm{dev}\, \mathrm{sym}\, p\|^2\,, &\quad W_{\mathrm{curl}}^{\mathrm{lin}}(\mathrm{Curl}\, p) = \frac{\mu\, L_c^2}{2} \|\, \mathrm{Curl}\, p\|^2\,, \end{aligned} \tag{2}$$

where $\mu, \lambda > 0$ are the Lamé constants, h^+ is the local hardening modulus and L_c with dimension length sets the plastic length scale. Note that the **infinitesimal plastic distortion** $p : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{M}^{3\times 3}$ need **not** be **symmetric**, but that only its symmetric part, the **infinitesimal plastic strain** $\mathrm{sym}\, p$, contributes to the local elastic energy expression. The **infinitesimal plastic rotation** $\mathrm{skew}\, p$ does not locally contribute to the elastic energy nor to the local plastic self-hardening but appears implicitly in the nonlocal hardening. The resulting elastic energy is invariant under infinitesimal rigid rotations $\nabla u \mapsto \nabla u + \overline{A}$, $\overline{A} \in \mathfrak{so}(3)$ of the body. The invariance of the curvature contribution needs the homogeneity of the infinitesimal rotations.

The evolution equation for the plastic distortion p follows by taking the variational derivative of the energy (2), where, due to the nonlocal contribution $\mathrm{Curl}\, p$, the possibility of specifying boundary conditions on p arises.

* Corresponding author E-mail: neff@mathematik.tu-darmstadt.de, Phone: +49 6151 16 34 95, Fax: +49 6151 16 6027

2 The strong formulation of geometrically linear strain gradient plasticity

The infinitesimal strain gradient plasticity model reads then: find the displacement $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the plastic distortion $p : \Omega \mapsto \mathfrak{sl}(3)$ with

$$\begin{aligned} u \in H^1([0,T]; H_0^1(\Omega, \Gamma_D, \mathbb{R}^3)), \quad \operatorname{sym} p \in H^1([0,T]; L^2(\Omega, \mathfrak{sl}(3))), \\ \operatorname{Curl} p(t) \in L^2(\Omega, \mathbb{M}^{3\times 3}), \quad \operatorname{Curl}\operatorname{Curl} p(t) \in L^2(\Omega, \mathbb{M}^{3\times 3}), \end{aligned} \tag{3}$$

such that

$$\begin{aligned} \operatorname{Div}\sigma = -f, \quad \sigma = 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr}[\nabla u - p]\,1\!\!1, \\ \dot p \in \partial\chi(\Sigma^{\text{lin}}), \quad \Sigma^{\text{lin}} = \Sigma_{\text{e}}^{\text{lin}} + \Sigma_{\text{sh}}^{\text{lin}} + \Sigma_{\text{curl}}^{\text{lin}}, \\ \Sigma_{\text{e}}^{\text{lin}} = 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr}[\nabla u - p]1\!\!1 = \sigma, \\ \Sigma_{\text{sh}}^{\text{lin}} = -2\mu\, h^+ \operatorname{dev}\operatorname{sym} p, \quad \Sigma_{\text{curl}}^{\text{lin}} = -\mu\, L_c^2 \operatorname{Curl}(\operatorname{Curl} p), \\ u(x,t) = u_{\text{d}}(x), \quad p(x,t).\tau = p(x,0).\tau, \quad x \in \Gamma_D, \\ 0 = [\operatorname{Curl} p(x,t)].\tau, \quad x \in \partial\Omega \setminus \Gamma_D, \quad p(x,0) = p^0(x). \end{aligned} \tag{4}$$

Here, χ is the indicator-function of the elastic domain.

If $p^0 \in \operatorname{Sym}(3)$, then $\Sigma_{\text{curl}}^{\text{lin}} = -\mu\, L_c^2 \operatorname{inc}(\varepsilon_p)$, i.e., the plastic strain incompatibility drives the nonlocal hardening; moreover $\Sigma_{\text{curl}}^{\text{lin}}$ is **symmetric** provided p^0 is symmetric, contrary to the finite strain case. The mathematically suitable space for symmetric p is the classical Sobolev-space $H_{\text{curl}}(\Omega) := \{v \in L^2(\Omega),\ \operatorname{Curl} v \in L^2(\Omega)\}$.

If, on the contrary, $p^0(x) \notin \operatorname{Sym}(3)$, then the linearized model will also have a non-zero plastic spin. It is, therefore, the initial condition on the plastic distortion p which determines whether this model is spin-free or not.

Note that in the large scale limit $L_c \to 0$ we recover a classical elasto-plasticity model with local kinematic hardening. Observe also that the term $\mu\, L_c^2 \operatorname{Curl}(\operatorname{Curl} p)$ acts as **nonlocal kinematical backstress** and constitutes a crystallographically motivated alternative to merely phenomenologically motivated backstress tensors. The term $-2\mu\, h^+ \operatorname{dev}\operatorname{sym} p$ is a **symmetric local kinematical backstress**. The model is therefore able to represent linear kinematic hardening and Bauschinger-like phenomena. Moreover, the driving stress Σ^{lin} is non-symmetric due to the presence of the second order gradients, while the local contribution σ, due to elastic lattice strains, remains symmetric.

The infinitesimal local contributions are fully rotationally invariant (isotropic and objective) with respect to the transformation $(\nabla u, p) \mapsto (\nabla u + A(x), p + A(x))$ and the nonlocal dislocation potential is still invariant with respect to the infinitesimal rigid transformation $(\nabla u, p) \mapsto (\nabla u + \overline{A}, p + \overline{A})$.

In this present form, it can be shown that classical solutions to the system (4) are unique, see [1]. However, existence is not clear. Using the Legendre-transformation of the indicator function χ one can convert the problem into a quasivariational inequality [1]. Existence and uniqueness is then a matter of defining appropriate Hilbert-spaces in which coercivity can be established. In this respect, it is remarkable that

$$\int_\Omega \|\operatorname{sym} p\|^2 + \|\operatorname{Curl} p\|^2 \,\mathrm{dx} \tag{5}$$

defines a norm for the not necessarily symmetric plastic distortion p provided tangential homogeneous boundary conditions $p(x).\tau = 0$ are satisfied on Γ_D. This is the basis for the introduction of a new Hilbert space appropriate for dislocation based plasticity.

References

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