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Homogeneous Cauchy stress induced by non-homogeneous deformations

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We discuss whether homogeneous Cauchy stress implies homogeneous strain in isotropic nonlinear elasticity. While for linear elasticity the positive answer is clear, we exhibit an example with inhomogeneous continuous deformation but constant Cauchy stress. The example is derived from a non rank-one convex elastic energy.

1 Linear elastic deformations

The isotropic linear elastic energy takes the form $W_{\text{lin}}(\nabla u) = \mu\|\text{dev sym}\,\nabla u\|^2 + \frac{\kappa}{2}\left[\text{tr}(\text{sym}\,\nabla u)\right]^2$, where $\mu > 0$ is the shear and $\kappa > 0$ is the bulk modulus. The corresponding stress-strain law is $\sigma = 2\mu\,\text{dev}\,\varepsilon + \kappa\,\text{tr}(\varepsilon)\mathbb{1}$ with the infinitesimal strain tensor $\varepsilon = \text{sym}\,\nabla u$. This mapping is invertible if and only if $\mu > 0$ and $\kappa > 0$. In this case $\sigma^{-1} : \text{Sym}(3) \to \text{Sym}(3)$ exists. In addition, when the Cauchy stress $\sigma = \overline{T}$ is constant, the homogeneous displacement $u(X) = \left[\sigma^{-1}(\overline{T}) + \overline{A}\right] X + \overline{b}$ is uniquely determined up to infinitesimal rigid body rotations $\overline{A} \in \mathfrak{so}(3)$ and translations $\overline{b} \in \mathbb{R}^3$.

2 Nonlinear elastic deformations

In contrast to linear elasticity, in nonlinear elasticity many different stress tensors are employed, for example the first Piola-Kirchhoff stress $S_1 = D_F W(F)$ and the (true) Cauchy stress $\sigma = S_1(F) \cdot (\text{Cof}\,F)^{-1}$.
Now we want to answer the following questions: Does homogeneous Cauchy stress σ imply homogeneous strain and if not, how can a homogeneous Cauchy stress be generated by non-homogeneous finite-strain deformations?
We already know that homogeneous Cauchy stress implies a self-equilibrated field and homogeneous strain causes all stress tensors to be homogeneous. Moreover, for a homogeneous isotropic hyperelastic body under finite strain deformation, the Cauchy stress tensor takes the form $\sigma(B) = \beta_0\mathbb{1} + \beta_1 B + \beta_{-1}B^{-1}$, where $B = FF^T$ is the left Cauchy-Green tensor and

$$\beta_0 = \frac{2}{\sqrt{\text{I}_3}}\left(\text{I}_2\frac{\partial W}{\partial \text{I}_2} + \text{I}_3\frac{\partial W}{\partial \text{I}_3}\right), \quad \beta_1 = \frac{2}{\sqrt{\text{I}_3}}\frac{\partial W}{\partial \text{I}_1}, \quad \beta_{-1} = -2\sqrt{\text{I}_3}\frac{\partial W}{\partial \text{I}_2}$$

are scalar functions of the principal invariants

$$\text{I}_1(B) = \text{tr}\,B = \|F\|^2, \quad \text{I}_2(B) = \frac{1}{2}\left[(\text{tr}\,B)^2 - \text{tr}\,B^2\right] = \|\text{Cof}\,F\|^2, \quad \text{I}_3(B) = \det B = (\det F)^2,$$

with $W(\text{I}_1, \text{I}_2, \text{I}_3)$ as the strain energy density function describing the physical properties of the isotropic hyperelastic material. W should be stress free in the reference configuratin Ω, i.e. $\beta_0 + \beta_1 + \beta_{-1}|_{F=\mathbb{1}} = 0$.
If $\sigma : \text{Sym}^+(3) \to \text{Sym}(3)$ is invertible, then for constant Cauchy stress $\sigma = \overline{T}$ we have a unique left Cauchy-Green tensor $\overline{B} \in \text{Sym}^+(3)$ which satisfies $\nabla\varphi(\nabla\varphi)^T = \overline{B} = \sigma^{-1}(\overline{T})$. The latter implies (formally equivalent to the infinitesimal situation) that $\varphi(X) = \left(\overline{V}\,\overline{R}\right)X + \overline{b} = \left[\sqrt{\sigma^{-1}(\overline{T})}\,\overline{R}\right] X + \overline{b}$, where $\overline{R} \in \text{SO}(3)$ is an arbitrary constant rotation, $\overline{b} \in \mathbb{R}^3$ is an arbitrary constant translation, and $\overline{V}$ is the left principal stretch tensor satisfying $\overline{V}^2 = \overline{B}$, which is uniquely determined by the given Cauchy stress $\sigma = \overline{T}$.

2.1 New strain energy function

We define the strain energy function

$$W = \frac{\mu}{2}\left(\text{I}_3^{-1/3}\text{I}_1 - 3\right) + \frac{\tilde{\mu}}{4}\left(\text{I}_1 - 3\right)^2 + \frac{\kappa}{2}\left(\text{I}_3^{1/2} - 1\right)^2, \tag{1}$$

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where $\mu > 0$ is the infinitesimal shear modulus, $\kappa > 0$ is the infinitesimal bulk modulus, and $\tilde{\mu} > 0$ is a positive constant independent of the deformation. For this material, the coefficients β_0, β_1 and β_{-1} depend only on the principal invariants I_1 and I_3, and W is stress free in the reference configuration [1, 2]. Consider the rank-one compatible deformation gradients

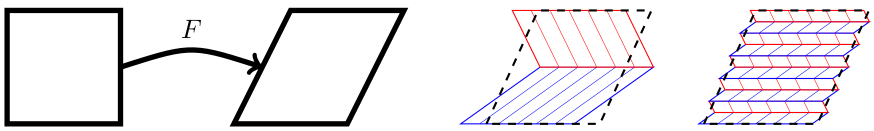
$$F = \begin{bmatrix} k & sa & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a \end{bmatrix}, \qquad \widehat{F} = \begin{bmatrix} k & -sa & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a \end{bmatrix},$$

where $k > 0$, $a > 0$, and $s > 0$ are positive constants. The corresponding left Cauchy-Green tensors B and $\widehat{B}$ have the same principal invariants and the associated Cauchy stress tensors are $\sigma(B) = \beta_0 \mathbb{1} + \beta_1 B$, $\sigma(\widehat{B}) = \beta_0 \mathbb{1} + \beta_1 \widehat{B}$.

Then, if $\frac{\mu}{3\tilde{\mu}} < \left(\frac{3-a^2-1/a^2}{4}\right)^{4/3}$ and $0 < s < \frac{1}{a}\sqrt{3 - 4\left(\frac{\mu}{3\tilde{\mu}}\right)^{3/4} - a^2 - \frac{1}{a^2}}$, there exists $k = k_0 \in (0, 1)$, such that $\beta_1 = 0$ and $\sigma(B) = \beta_0 \mathbb{1} = \sigma(\widehat{B})$. Thus, we obtain homogeneous Cauchy stress although we suppose a non-homogeneous deformation.

3 Constitutive requirements in nonlinear elasticity

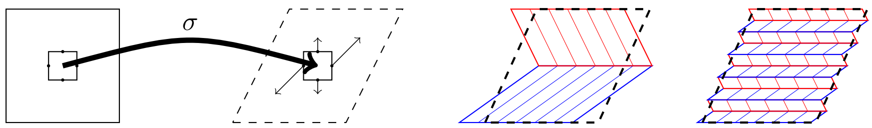
Under *homogeneous Dirichlet boundary conditions* $\varphi(x) = Fx$, quasiconvexity implies that only the homogeneous solution is energy optimal, whereas in the case of strict rank-one convexity **no (infinitesimal) rank-one laminate can be energy optimal**.



energy optimal homogeneous solution

a rank-one laminate is not energy optimal

When we consider *homogeneous interior Cauchy stress* $\sigma = const.$, strict rank-one convexity implies that **no rank-one laminate is in equilibrium**, while invertibility of σ induces that only the homogeneous solution is in equilibrium.



homogeneous solution is in equilibrium

a rank-one laminate is not in equilibrium

4 Invertibility of the Cauchy stress σ

Whereas invertibility of the first Piola-Kirchhoff stress S_1 violates material objectivity (frame-indifference), invertibility of the Cauchy stress tensor σ is not in conflict with any known physical principle, and therefore it may be imposed as a constitutive requirement. For example consider the *exponentiated Hencky-type energy*

$$\begin{aligned} W_{eH}(\log V) &= \frac{\mu}{\kappa}\, e^{\|\mathrm{dev}_3 \log V\|^2} + \frac{\kappa}{2\widehat{\kappa}}\, e^{\widehat{\kappa}\,[\mathrm{tr}(\log V)]^2}, \qquad (2) \\ \sigma_{eH}(\log V) &= 2\mu\, e^{\kappa\, \|\mathrm{dev}_3 \log V\|^2 - \mathrm{tr}(\log V)} \cdot \mathrm{dev}_3 \log V + \kappa\, e^{\widehat{\kappa}\,[\mathrm{tr}(\log V)]^2 - \mathrm{tr}(\log V)}\ \mathrm{tr}(\log V) \cdot \mathbb{1}, \end{aligned}$$

where $V = \sqrt{FF^T}$ is the left stretch tensor. Then σ_{eH} is invertible, while W_{eH} is not rank-one convex [3–5].

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