PAMM · Proc. Appl. Math. Mech. **16**, 341 – 342 (2016) / **DOI** 10.1002/pamm.201610159

Loss of ellipticity in additive logarithmic finite strain plasticity and related results on Hencky-type energies

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The aim of this paper is to present some results regarding the Legendre-Hadamard ellipticity and loss of ellipticity of some energies depending on the logarithmic strain tensor.

1 Why a new Hencky-type energy?

We consider nonlinear elastic energies based on certain invariants of the logarithmic strain tensor $\log U$, namely $\|\operatorname{dev}_n \log U\|^2$ and $[\operatorname{tr}(\log U)]^2$, where $F = \nabla\varphi$ is the deformation gradient, $U = \sqrt{F^T F}$ is the right stretch tensor, $\operatorname{dev}_n X = X - \frac{1}{n}\operatorname{tr}(X)\, \mathbb{1}_n$ is the deviatoric part of the second order tensor $X \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ is the Frobenius tensor norm.

The fact that the quadratic Hencky energy $W_{\rm H}(F) = \mu\|\operatorname{dev}_n \log U\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2$ represents a useful tool in nonlinear elasticity is a common place [5]. Here $\mu > 0$ is the shear (distortional) modulus, $\kappa = \frac{2\mu+3\lambda}{3} > 0$ is the bulk modulus with the first Lamé constant λ. However, it is also well known [4] that the quadratic Hencky energy $W_{\rm H}$ is not Legendre-Hadamard elliptic, for $n = 2, 3$ and for any $\mu, \kappa > 0$. Motivated by this serious shortcoming, in some previous works [3,6–9] we have considered the following family of exponentiated Hencky-logarithmic strain type energies

$$W_{\rm eH} : \mathrm{GL}^+(n) \to \mathbb{R}, \qquad W_{\rm eH}(F) = \frac{\mu}{k}\, e^{k\,\|\operatorname{dev}_n \log U\|^2} + \frac{\kappa}{2\,\widehat{k}}\, e^{\widehat{k}\,(\operatorname{tr}(\log U))^2},$$

where $k, \widehat{k}$ are dimensionless parameters. These energies approximate the classical quadratic Hencky strain energy $W_{\rm H}$ for deformation gradients F sufficiently close to the identity $\mathbb{1}_n$.

In the planar case ($\operatorname{dev}_2 \log U = \log U - \frac{1}{2}\operatorname{tr}(\log U)\, \mathbb{1}_2$), the exponentiated Hencky energies $W_{\rm eH}$ are polyconvex [3, 9] for $\mu > 0, \kappa > 0$, $k \geq \dfrac{1}{4}$ and $\widehat{k} \geq \dfrac{1}{8}$. Since polyconvexity of an energy implies its Legendre-Hadamard ellipticity [1], for $n = 2$ the exponentiated Hencky energy avoid the above mentioned shortcoming of the quadratic Hencky energy.

2 A finite strain multiplicative plasticity formulation

In the three-dimensional case, it is established [8] that for all $k > 0$, the function $F \mapsto e^{k\,\|\operatorname{dev}_3 \log U\|^2}$, $F \in \mathrm{GL}^+(3)$ is not Legendre-Hadamard elliptic. However, we discuss an interesting relation between non-ellipticity of $W_{\rm eH}$ in three-dimensions and finite plasticity models. Using the results from [2] we

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obtain that the ellipticity domain of the isochoric energy part $W_{\text{eH}}^{\text{iso}}(F) = e^{k\,\|\text{dev}_3 \log U\|^2}$ contains the domain $\left\{U \in \text{Sym}^+(3) \;\middle|\; \|\text{dev}_3 \log U\|^2 \le \frac{2}{3}\right\}$ which is related to the von-Mises-Huber-Hencky criterion (maximum distortiontowith $2\,\mu e^{k\frac{2}{3}}$.

Thus, we turn to finite strain isotropic plasticity [6] and we couple the exponentiated Hencky energy with a finite strain multiplicative plasticity formulation. Multiplicative plasticity is stable at frozen plastic flow, i.e. if the elastic energy $F \mapsto W(F)$ is Legendre-Hadamard elliptic, it follows that the elasto-plastic formulation $F \mapsto W(F, F_p) := W(F\,F_p^{-1})$ remains Legendre-Hadamard elliptic w.r.t. F for all given plastic distortions F_p. Hence, in the multiplicative setting, the elastic rank-one convexity is independent of the plastic flow. Therefore, the multiplicative approach is ideally suited as far as preservation of ellipticity properties for elastic unloading is concerned and marks a sharp contrast to the additive logarithmic modelling frameworks, as it will be seen.

3 Loss of ellipticity in additive logarithmic finite strain plasticity

Another finite plasticity model using logarithmic strains is taking the additive elastic Hencky energy in the format $\widehat{W}_{\text{H}}(\log U - \log U_p) = \mu\,\|\text{dev}_3[\log U - \log U_p]\|^2 + \frac{\kappa}{2}\,[\text{tr}(\log U)]^2$, as a starting point, in which plastic incompressibility $\det U_p = 1$ is already included. Regarding the additive logarithmic finite strain plasticity, it is established [7] that:

Proposition 3.1 *The function* $F \mapsto W(F) = e^{\|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2}$ *is not Legendre-Hadamard elliptic for some given plastic stretch* $U_p \in \text{Sym}^+(2)$*, while* $F \mapsto W(F) = e^{\|\text{dev}_2 \log U\|^2}$ *is Legendre-Hadamard elliptic.*

Note that the loss of ellipticity occurs for the given plastic strain U_p, at elastic strains in the order of $\|\text{dev}_3 \log U_p\| \approx 2\sqrt{2}$ which means 382% elastic strain. The conclusion of this section and the main conclusion of this note is that the additive logarithmic model does not preserve Legendre-Hadamard ellipticity in elastic unloading in general.

Acknowledgements The work of the first author was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-RU-TE-2014-4-1109.

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