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Null-Lagrangians and the indeterminate couple stress model

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The aims of this note is to present a new model based on a new representation of the curvature energy in the indeterminate couple stress model and to discuss some related choices from the literature.

1 Introduction

We consider the second gradient elasticity energy $W = \mu \,\|\mathrm{sym}\nabla u\|^2 + \frac{\lambda}{2}\,[\mathrm{tr}(\mathrm{sym}\nabla u)]^2 + W_{\mathrm{curv}}(D^2u)$, where $W_{\mathrm{curv}}(D^2u)$ admits the following representations in the indeterminate couple stress theory [1, 4]

$$\begin{aligned} W_{\mathrm{curv}}(D^2u) =&\ \mu\, L_c^2\,[\alpha_1\,\|\,\mathrm{dev\,sym}\,\nabla[\mathrm{axl}(\mathrm{skew}\nabla u)]\|^2 + \alpha_2\,\|\mathrm{skew}\nabla[\mathrm{axl}(\mathrm{skew}\nabla u)]\|^2] \quad \text{classical}\\ =&\ \mu\, L_c^2\,[\alpha_1\,\|\,\mathrm{dev\,sym\,Curl}(\mathrm{sym}\,\nabla u)\|^2 + \alpha_2\,\|\mathrm{skew\,Curl}(\mathrm{sym}\,\nabla u)\|^2] \quad \text{new.}\end{aligned}$$

Here, we have used the identity $\underbrace{\nabla[\mathrm{axl}(\mathrm{skew}\nabla u)]}_{\text{rotation gradient}} = \underbrace{[\mathrm{Curl}(\mathrm{sym}\ \nabla u)]^T}_{\text{strain gradient}}$, which allows us easily to switch from considerations on the level of strain gradients to the level of rotational gradients and vice versa. For notations we refer the reader to [1, 3, 5]

2 A new formulation of the indeterminate couple stress model

Using the above equivalent forms of the curvature energy, we obtain the balance equations

$$\mathrm{Div}(\sigma - \widetilde{\tau}) + f = 0 \quad \text{(classical)} \qquad \text{versus} \qquad \mathrm{Div}(\sigma + \widehat{\tau}) + f = 0 \quad \text{(new model)}$$

where

$$\begin{aligned} &\widetilde{\tau} = \frac{1}{2}\mathrm{anti}(\,\mathrm{Div}\,\widetilde{m}) \in \mathfrak{so}(3) \qquad \text{versus} \qquad \widehat{\tau} = \mathrm{sym\,Curl}(\widehat{m}) \in \mathrm{Sym}(3),\\ &\sigma = 2\,\mu\ \mathrm{sym}\,\nabla u + \lambda\,\mathrm{tr}(\nabla u)\,\mathbb{1} \in \mathrm{Sym}(3),\\ &\widetilde{m} = \mu\,L_c^2\,[2\,\alpha_1\,\mathrm{dev\,sym}\,\nabla[\mathrm{axl}(\mathrm{skew}\nabla u)] + 2\,\alpha_2\,\mathrm{skew}\nabla[\mathrm{axl}(\mathrm{skew}\nabla u)]] \in \mathbb{R}^{3\times3},\\ &\widehat{m} = \mu\,L_c^2\,\{2\,\alpha_1\,\mathrm{dev\,sym}[\mathrm{Curl}(\mathrm{sym}\,\nabla u)] + 2\,\alpha_2\,\mathrm{skew}[\mathrm{Curl}(\mathrm{sym}\,\nabla u)]\} \ \in \mathbb{R}^{3\times3}.\end{aligned}$$

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Here, we observe that $\operatorname{Div}(\widetilde{\tau} + \widehat{\tau}) = 0$, i.e. the difference of both formulations resides in a term not visible in the balance equations, but only visible at the boundary. It appears through the way partial integration is performed as suggested (and not dictated) by the respective energy terms. The new model yields a symmetric total force stress tensor $\sigma + \widehat{\tau}$ and if $\alpha_2 = 0$ also a symmetric, trace free moment stress tensor $\widehat{m}$ with a least number of material parameters. It is conformally invariant and well-posed. In contrast, the classical indeterminate couple stress model has a non-symmetric total force stress tensor $\sigma - \widetilde{\tau}$. This development shows that one may always fall back to a **symmetric total force stress tensor** (Boltzmann-axiom) and symmetric moment stress tensors at the expense of dealing with modified boundary conditions. What, then, is the meaning of the symmetry or non-symmetry of stress tensors in these higher gradient linear elasticity models? We surmise that the symmetry condition for the total stress tensor can be used to fix the physically unknown higher order traction boundary conditions. In that sense, the symmetry requirement acts like a gauge condition fixing one preferred possibility. There are also some possible boundary conditions which arise naturally in the indeterminate couple stress theory when viewed from the perspective of the full gradient model [3] and which are not the same as in the direct approach of Mindlin and Tiersten [4].

3 Conclusion

The new variant of the indeterminate couple stress theory shows that the way chosen in order to obtain the equilibrium equations is a constitutive choice, a modelling choice.

All the differences between the various alternative couple stress models can be traced back to the appearance of Null-Lagrangians either on the level of the total stresses or on the level of the moment stresses. Null-Lagrangians leave the Euler-Lagrange equations invariant but alter the boundary conditions. For example, the Grioli's Null-Lagrangian [2] $\langle \nabla \operatorname{curl} u, (\nabla \operatorname{curl} u)^T \rangle = \operatorname{tr}[(\nabla \operatorname{curl} u)^2] = [\operatorname{tr}(\nabla \operatorname{curl} u)]^2 - 2\operatorname{tr}[\operatorname{Cof}(\nabla \operatorname{curl} u)] = \operatorname{div}[Z(\operatorname{curl} u, \nabla \operatorname{curl} u)]$ modifies only the format of the moment stresses, since $\frac{\partial \langle \nabla \operatorname{curl} u, (\nabla \operatorname{curl} u)^T \rangle}{\nabla \operatorname{curl} u} = [\nabla \operatorname{curl} u]^T = 2[\nabla \operatorname{axl}(\operatorname{skew} \nabla u)]^T$ is added on the Mindlin's moment stresses, while, since $\operatorname{anti}\operatorname{Div}[\nabla \operatorname{axl}(\operatorname{skew} \nabla u)]^T = \operatorname{anti}\operatorname{Div}[\operatorname{Curl}(\operatorname{sym} \nabla u)] = 0$, the total force stress tensor $\sigma - \widetilde{\tau}$ is not changed. Therefore, the Grioli's Null-Lagrangian has no contribution to the balance equation. For each choice of Null-Lagrangian different "material parameters" intervene. The question arises: how can we identify boundary value problem-independent parameters in the indeterminate couple stress model when the only effect different representations have is on the boundary conditions? We believe that this is one of the fundamental unresolved issues in linear, isotropic gradient elasticity. Every real advance in the subject will be connected to choosing the "right" Null-Lagrangian.

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