Numerical results for an elasto-plastic Cosserat model with more than one million dof in 3D

Wolfgang Müller[*1], **Patrizio Neff**[**2], and **Christian Wieners**[***1]

[1] Fakultät für Mathematik, Universität Karlsruhe (TH), Kaiserstr. 12, 76128 Karlsruhe, Germany.
[2] Fachbereich Mathematik, Darmstadt Technische Universität, Schlossgartenstrasse 7, 64289 Darmstadt, Germany.

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We present a finite element implementation of a infinitesimal elasto-plastic Cosserat model allowing for non-symmetric stresses. We use the von Mises yield criterion and restrict plastic effects to the classical elastic response as in the perfect plastic case. We review the nonlinear equations for the weak formulation of the problem. Numerical results are presented for 3D example with more than one million degrees of freedom.

In the geometrically linear generalized continua of Cosserat micropolar type we postulate independent infinitesimal micro-rotations of the material. Thus, as a consequence of balance of angular momentum, Cauchy stresses $\boldsymbol{\sigma}$ are not symmetric any more. It is known that Cosserat effects regularize the mesh size dependence of localization computations where shear failure mechanisms play a dominant role (see also [1]). Let $\Gamma_D \cup \Gamma_N \subset \partial\Omega$. We want to determine

displacements	$\mathbf{u}$:	$\overline{\Omega} \times [0,T] \longrightarrow \mathbf{R}^3$,
infinitesimal micro-rotations	$\overline{A}$:	$\Omega \times [0,T] \longrightarrow \mathfrak{so}(3)$,
non-symmetric stresses	$\boldsymbol{\sigma}$:	$\Omega \times [0,T] \longrightarrow \mathbf{R}^{3,3}$,
symmetric plastic strains	$\boldsymbol{\varepsilon}_p$:	$\Omega \times [0,T] \longrightarrow \mathrm{Sym}(3)$ with $\boldsymbol{\varepsilon}_p(0) = \mathbf{0}$,
and a plastic multiplier	Λ:	$\Omega \times [0,T] \longrightarrow \mathbf{R}$,

satisfying the essential boundary conditions and the equilibrium equations

$$\begin{array}{rcll} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t) &=& \mathbf{b}(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T]\ , \\ \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}(\mathbf{x}) &=& \mathbf{t}_N(\mathbf{x},t), & (\mathbf{x},t) \in \Gamma_N \times [0,T]\ , \\ -\mu L_c^2 \Delta \overline{A}(\mathbf{x},t) &=& \mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x},t)) - \overline{A}(\mathbf{x},t)\big), & (\mathbf{x},t) \in \Omega \times [0,T]\ , \\ D\overline{A}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) &=& \mathbf{0}, & (\mathbf{x},t) \in \Gamma_N \times [0,T]\ , \end{array}$$

the constitutive relation $\boldsymbol{\sigma} = 2\mu\big(\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p\big) + \lambda \operatorname{tr} D\mathbf{u}\cdot\mathbf{I} + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}) - \overline{A}\big)$, the complementary conditions for the yield criterion $\Lambda\phi(T_E) = 0$, $\Lambda \geq 0$, $\phi(T_E) \leq 0$, and the flow rule $\dot{\boldsymbol{\varepsilon}}_p = \Lambda D\phi(T_E)$, depending on $T_E = 2\mu\big(\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p\big)$, where we have neglected the arguments in space and time. The elasto-plastic Cosserat problem is well-posed [2, 3].

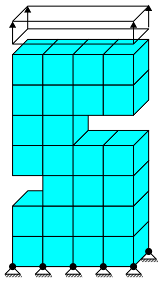
Let $\mathbf{V}_h \times W_h$ be a finite element space with $\mathbf{v}_h = \mathbf{0}$ and $\overline{B}_h = 0$ on Γ_D. The model of incremental infinitesimal plasticity is obtained by a decomposition in time and use of backward Euler scheme. The fully discrete elasto-plastic problem is equivalent to the following nonlinear weak problem [4]. For given $\boldsymbol{\varepsilon}_p^{n-1}$ find $(\mathbf{u}^n, \overline{A}^n) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} &\int_\Omega P_{\mathbf{K}}\big(2\mu(\operatorname{sym}(D\mathbf{u}_h^n) - \boldsymbol{\varepsilon}_p^{n-1})\big) : D\mathbf{v}_h\,d\mathbf{x} + \lambda \int_\Omega \operatorname{tr} D\mathbf{u}_h^n \cdot \operatorname{tr} D\mathbf{v}_h\,d\mathbf{x} \\ &\qquad + 2\mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}_h^n) - \overline{A}_h^n\big) : D\mathbf{v}_h\,d\mathbf{x} = \ell(t_n, \mathbf{v}_h), \qquad \mathbf{v}_h \in \mathbf{V}_h\ , \\ &\mu L_c^2 \int_\Omega D\overline{A}_h^n \cdot D\overline{B}_h\,d\mathbf{x} = \mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}_h^n) - \overline{A}_h^n\big) : \overline{B}_h\,d\mathbf{x}, \qquad \overline{B}_h \in W_h\ , \end{aligned}$$

with the orthogonal projection $P_{\mathbf{K}}(\boldsymbol{\theta}) = \boldsymbol{\theta} - \max\big\{0, |\operatorname{dev}(\boldsymbol{\theta})| - K_0\big\}\frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}$ on the elastic domain $\mathbf{K} := \{\boldsymbol{\tau} \in \mathbf{R}^{3,3} : \boldsymbol{\tau}^T = \boldsymbol{\tau}\ , |\operatorname{dev}\boldsymbol{\tau}| \leq K_0\}$ of the von Mises flow rule. For smooth Dirichlet data and smooth domain the solution of the time-discrete update problem is H^2-smooth [5]. The nonlinear weak problem can be written as a minimization problem [4].

For linear Lagrange elements, we have linear convergence in h. The elasto-plastic Cosserat Model is a regularization for classical perfect plasticity (shear failure mechanisms). Future work will be the analysis and robust implementation of geometrically nonlinear elasto-plastic Cosserat Models.

* e-mail: mueller@math.uni-karlsruhe.de, Phone: +49 721 608 3712, Fax: +49 721 608 3197
** e-mail: neff@mathematik.uni-darmstadt.de
*** e-mail: wieners@math.uni-karlsruhe.de



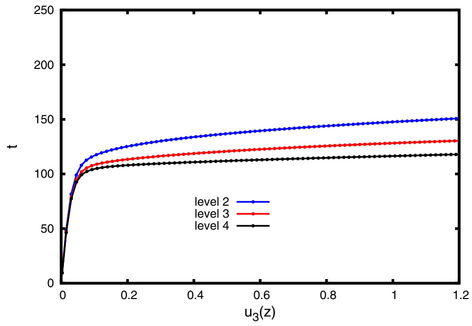


Fig. 1 Boundary conditions and coarse grid for notched cube. We compute load-displacement curve on uniform refinement level 2,3 and 4 ($\mu_c = 10^{-4}\mu$, $L_c = 10^{-2}$). On level 4 we compute up to 1.311.751 dof.

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