Neff, P.

On a viscoplastic approximation in finite plasticity

We are concerned with a phenomenological model of isotropic finite elasto-plasticity valid for small elastic strains applied to polycrystalline material. We prove a local in time existence and uniqueness result. To the best of our knowledge this is the first rigorous result concerning classical solutions in geometric nonlinear finite visco-plasticity.

1. Formulation of the problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded region with smooth boundary $\partial\Omega$. In the nonlinear theory of elasto-visco-plasticity at large deformation gradients it is often assumed that the deformation gradient $F = \nabla u$ splits multiplicatively into an elastic and plastic part

$$\nabla u(x) = F(x) = F_e(x) \cdot F_p(x), \quad F_e, F_p \in GL(3, \mathbb{R}) \tag{1}$$

where F_e, F_p are explicitly understood to be incompatible configurations. In the quasi-static visco-plastic setting without body forces we are led to study the following system of coupled partial differential and evolution equations for the deformation $u : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$, the plastic variable $F_p : [0,T] \times \overline{\Omega} \mapsto SL(3, \mathbb{R})$ and the elastic rotation $R_e : [0,T] \times \overline{\Omega} \mapsto SO(3)$

$$\begin{aligned}
0 &= \operatorname{div} D_F\left[W(F_e, R_e)\right] = \operatorname{div} S_1(F_e, R_e) \\
W(F_e, R_e) &= \frac{\mu}{4}\, \|F_e^T \cdot R_e + R_e^T \cdot F_e - 2\mathbb{1}\|^2 + \frac{\lambda}{8}\, \mathrm{tr}(\mathrm{F_e^T} \cdot \mathrm{R_e} + \mathrm{R_e^T} \cdot \mathrm{F_e} - 2\mathbb{1})^2 \\
F_e &= \nabla u \cdot F_p^{-1} \\
S_1(F_e, R_e) &= R_e \left[\mu(F_e^T R_e + R_e^T F_e - 2\mathbb{1}) + \lambda\, \mathrm{tr}(\mathrm{F_e^T R_e} - \mathbb{1})\right] F_p^{-T} \\
u_{|\partial\Omega}(t, x) &= g(t, x), \quad x \in \partial\Omega \\
\frac{d}{dt}\left[F_p^{-1}\right](t) &= -F_p^{-1}(t) \cdot \partial\chi(\Sigma_M) \\
\Sigma_M &= F_e^T \cdot D_{F_e} W(F_e, R_e) \\
&= F_e^T R_e \left[\mu(F_e^T R_e + R_e^T F_e - 2\mathbb{1}) + \lambda\, \mathrm{tr}(\mathrm{F_e^T R_e} - \mathbb{1}) \cdot \mathbb{1}\right] \\
F_p^{-1}(0) &= F_{p_0}^{-1}, \qquad \det F_{p_0} = 1, \quad F_{p_0} \in SL(3) \\
\frac{d}{dt} R_e(t) &= \nu^+ \cdot \mathrm{skew}\,(B) \cdot R_e(t) \\
B &= B_{exact} \quad or \quad B_{approx.} \\
B_{exact} &= F_e R_e^T \\
B_{approx.} &= \left[\mu(2 \cdot \mathbb{1} - F_e R_e^T) + 2\lambda\left[3 - \langle F_e R_e^T, \mathbb{1}\rangle\right]\right] \cdot F_e R_e^T \\
\nu^+ &= \nu^+(F_e, R_e) \in \mathbb{R} \\
R_e(0) &= R_e^0, \qquad R_e^0 \in SO(3)
\end{aligned} \tag{2}$$

with a nonlinear flow potential $\chi : \mathrm{M}^{3\times 3} \mapsto \mathbb{R}$ that governs the visco-plastic evolution and which is motivated through the principle of maximal dissipation relevant for the thermodynamical consistency of the model. Here Σ_M denotes the so called Mandel stress tensor and S_1 is the first Piola-Kirchhoff stress. The term $\nu^+ := \nu^+(F_e, R_e)$ represents a scalar valued penalty function introducing viscosity. $F_{p_0}^{-1}$ and R_e^0 are the initial conditions for the plastic variable and elastic rotation part, respectively. For consistency one should put $R_e^0 = \overline{R} \cdot polar(F_{p_0}^{-1})$ with $\overline{R}$ any rigid rotation describing a prerotated intermediate configuration. The Lamé constants μ, λ of the polycrystalline material under consideration are assumed to be non-negative throughout with $\mu > 0$.

In the limit $\nu^+ \to \infty$ the model (2) approaches formally the problem

$$\begin{aligned} 0 &= \operatorname{div} D_F\left[W_\infty(F_e)\right] \\ W_\infty(F_e) &= \mu\,\|U_e-1\!\!1\|^2+\frac{\lambda}{2}\operatorname{tr}(U_e-1\!\!1)^2 \\ \frac{d}{dt}\left[F_p^{-1}\right](t) &= -F_p^{-1}(t)\cdot\partial\chi(\Sigma_{M,\infty}) \\ \Sigma_{M,\infty} &= U_e\cdot D_{U_e}W(U_e)=U_e\left[2\mu(U_e-1\!\!1)+\lambda\operatorname{tr}(U_e-1\!\!1)\cdot 1\!\!1\right] \end{aligned} \tag{3}$$

with $U_e=(F_e^TF_e)^{\frac{1}{2}}$ and $U_e-1\!\!1$ the elastic Biot strain tensor. The system (3) is an exact model for small elastic strains and finite plastic deformations. For a detailed account on the properties of the model (2) we refer the interested reader to the original paper.

We can prove the following result

Theorem 1. *Suppose for the displacement boundary data $g\in C^1(\mathbb{R},H^{5,2}(\Omega,\mathbb{R}^3))$. Then there exists a time $t_1>0$ such that the initial boundary value problem (2) in a viscous form admits a unique solution*

$$(u,F_p,R_e)\in C([0,t_1],H^{5,2}(\Omega,\mathbb{R}^3))\times C^1([0,t_1],H^{4,2}(\Omega,SL(3,\mathbb{R})),H^{4,2}(\Omega,SO(3))). \tag{4}$$

Idea of the proof: at frozen variables (F_p,R_e) the above (elastic) equilibrium system proves to be a linear, second order, strictly Legendre-Hadamard elliptic boundary value problem with nonconstant coefficients. This system has variational structure in the sense that the equilibrium part of (2) is formally equivalent to the minimization problem

$$\begin{aligned} \forall\, t\in[0,T] &: \quad I(u(t),F_p^{-1}(t),R_e(t))\mapsto min,\quad u(t)\in g(t)+H_\circ^{1,2} \\ I(u,F_p^{-1},R_e) &= \int_\Omega W(\nabla u\cdot F_p^{-1},R_e)\,dx \\ W(F_e,R_e) &= \frac{\mu}{4}\,\|F_e^T\cdot R_e+R_e^T\cdot F_e-2 1\!\!1\|^2+\frac{\lambda}{8}\operatorname{tr}(F_e^T\cdot R_e+R_e^T\cdot F_e-2 1\!\!1)^2. \end{aligned} \tag{5}$$

The main task in proving that (2) is well posed consists of showing uniform estimates for solutions of elliptic systems whose coefficients are time dependent and do not induce a pointwise positive bilinear form. This problem does not arise in infinitesimal elasto-viscoplasticity since there the elasticity tensor is assumed to be a constant positive definite fourth order tensor. We are first concerned with the static situation where (F_p,R_e) are assumed to be known. We prove the existence, uniqueness and regularity of solutions to the related (elastic) boundary value problem. In addition we elucidate in which manner these solutions depend on (F_p,R_e). These investigations rely heavily on a theorem recently proved by the author extending Korns first inequality to nonconstant coefficients.

The conceptual idea to treat the evolution problem is straightforward: we write the ordinary differential equation in the following form

$$\frac{d}{dt}A(t)=h(\nabla_x u(A),A)\cdot A \tag{6}$$

where $A=(F_p^{-T},R_e)$ and $u=u(A)$ is the solution of the (elastic) elliptic boundary value problem at given A. It remains to show that the right hand side as a function of A is locally Lipschitz in some properly defined Banach space allowing to apply the well known local existence and uniqueness theorem.

2. References

1 Neff, P.: Formulation of visco-plastic approximations in finite plasticity for a model of small elastic strains. Part I: Modelling. Preprint TU Darmstadt No. 2127 (2000).
2 Neff, P.: Formulation of visco-plastic approximations in finite plasticity for a model of small elastic strains. Part IIa: Local existence und uniqueness results. Preprint TU Darmstadt No. 2138 (2001).
3 Neff, P.: On Korns first inequality with nonconstant coefficients; to appear in Proc. Roy. Soc. Edinb. Ser. A.

Dr. Patrizio Neff, Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstrasse 7, 64289 Darmstadt, Germany