On isotropy conditions in second gradient materials

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In gradient elasticity, isotropy and frame-indifference requirements are sensitive to the homogeneity of the applied rotation field $Q \in \mathrm{SO}(3)$. This is in contrast to standard elasticity, where only first gradients of the deformation are under consideration. We use a diffeomorphism to show the effect of inhomogeneous coordinate transformation to the form-invariance requirement of elastic energy. From a classical geometric rigidity result follows that the appearance of a right-local $\mathrm{SO}(3)$-invariance condition is not the general condition for isotropy. The correct statement for isotropy in second gradient elasticity should be a right-global $\mathrm{SO}(3)$-invariance condition.

1 Global versus local rotational invariance for isotropy

In hyperelasticity, the difference between form-invariance under compatible transformations of the reference configuration with rigid rotations $\overline{Q}$ (isotropy) and right-invariance under inhomogeneous rotation fields $Q = Q(x) \in \mathrm{SO}(3)$ becomes visible only in higher gradient elasticity. To see this, consider coordinates $x \in \mathbb{R}^3$ transformed to $\xi \in \mathbb{R}^3$ via the diffeomorphism $\zeta : B \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$

$$x = \zeta(\xi)\,, \qquad \xi = \zeta^{-1}(x)\,, \qquad x = \zeta(\zeta^{-1}(x))\,, \tag{1}$$

see also [1,2]. Connected to the coordinate transformation (1) we consider the deformation expressed in these new coordinates via setting

$$\varphi^\flat(\xi) := \varphi(\zeta(\xi))\,, \qquad \varphi^\flat(\zeta^{-1}(x)) = \varphi(x)\,. \tag{2}$$

Let the elastic energy of the body $B \subset \mathbb{R}^3$ depend on first and second gradients of the deformation $\varphi(x)$. We say that the elastic energy is **form-invariant** with respect to the (referential) coordinate transformation ζ if and only if

$$\int\limits_{x \in B} W\left(\mathrm{Grad}_x[\varphi(x)],\, \mathrm{GRAD}_x[\mathrm{Grad}_x[\varphi(x)]]\right)\, \mathrm{d}x = \int\limits_{\xi \in \zeta^{-1}(B)} W\Big(\mathrm{Grad}_\xi[\varphi^\flat(\xi)],\, \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\varphi^\flat(\xi)]]\Big)\, \mathrm{d}\xi\,. \tag{3}$$

For the first and second derivative with respect to x we obtain from eq.(1)

$$\mathbb{1} = \mathrm{Grad}_\xi[\zeta(\zeta^{-1}(x))]\, \mathrm{Grad}_x[\zeta^{-1}(x)] \quad \Leftrightarrow \quad \mathrm{Grad}_x[\zeta^{-1}(x)] = (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1}\,, \tag{4}$$

and

$$\mathrm{Grad}_\xi[\zeta(\zeta^{-1}(x))]\, \mathrm{GRAD}_x[\mathrm{Grad}_x[\zeta^{-1}(x)]] = -\mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\zeta(\xi)]]\, \mathrm{Grad}_x[\zeta^{-1}(x)]\, \mathrm{Grad}_x[\zeta^{-1}(x)]\,, \tag{5}$$

yielding

$$\mathrm{GRAD}_x[\mathrm{Grad}_x[\zeta^{-1}(x)]] = -(\mathrm{Grad}_\xi[\zeta(\xi)])^{-1}\, \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\zeta(\xi)]]\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1}\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1} \tag{6}$$

Thus, (3) is **form-invariant** with respect to the (referential) coordinate transformation ζ if and only if

$$\begin{aligned}
&\int\limits_{\xi \in \zeta^{-1}(B)} W\Big(\mathrm{Grad}_\xi[\varphi^\flat(\xi)]\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1},\, \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\varphi^\flat(\xi)]]\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1}\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1} \\
&\quad - \mathrm{Grad}_\xi[\varphi^\flat(\xi)]\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1}\, \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\zeta(\xi)]]\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1}\, (\mathrm{Grad}_\xi[\zeta(\xi)])^{-1}\Big) \\
&\qquad \det(\mathrm{Grad}_\xi[\zeta(\xi)])\, \mathrm{d}\xi \\
&\qquad\qquad = \int\limits_{\xi \in \zeta^{-1}(B)} W\Big(\mathrm{Grad}_\xi[\varphi^\flat(\xi)],\, \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\varphi^\flat(\xi)]]\Big)\, \mathrm{d}\xi\,.
\end{aligned} \tag{7}$$

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Equality (7) can be specified to

$$\det(\mathrm{Grad}_\xi[\zeta(\xi)]) = 1\,, \quad \mathrm{Grad}_\xi[\zeta(\xi)] \in \mathcal{G} \subseteq \mathrm{SO}(3) \qquad \forall\, \xi \in \zeta^{-1}(B)\,, \tag{8}$$

where $\mathcal{G}$ is the symmetry group of the material. We set $(\mathrm{Grad}_\xi[\zeta(\xi)])^{-1} = Q(\xi)$ and obtain as first concise form-invariance statement for material symmetry

$$\begin{aligned}\int\limits_{\xi\,\in\,\zeta^{-1}(B)} W\Big(&\mathrm{Grad}_\xi[\varphi^\flat(\xi)]\, Q(\xi),\ \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\varphi^\flat(\xi)]]\, Q(\xi)\, Q(\xi)\\ &- \mathrm{Grad}_\xi[\varphi^\flat(\xi)]\, Q(\xi)\, \mathrm{GRAD}_\xi[Q^T(\xi)]\, Q(\xi)\, Q(\xi)\Big)\, 1\ \mathrm{d}\xi\\ &= \int\limits_{\xi\,\in\,\zeta^{-1}(B)} W\Big(\mathrm{Grad}_\xi[\varphi^\flat(\xi)],\ \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\varphi^\flat(\xi)]]\Big)\ \mathrm{d}\xi \quad \forall Q(\xi) \in \mathcal{G}\,,\end{aligned} \tag{9}$$

which we will call **right-local SO(3)-invariance** since the rotations in eq.(9) are allowed to be inhomogeneous. However, requiring that

$$(\mathrm{Grad}_\xi[\zeta(\xi)])^{-1} = Q^T(\xi) \in \mathrm{SO}(3) \quad \Leftrightarrow \quad \mathrm{Grad}_\xi[\zeta(\xi)] = Q(\xi)\ \in \mathrm{SO}(3) \tag{10}$$

means, by a **classical geometric rigidity** result, see e.g. [3], that

$$\mathrm{Grad}_\xi[\zeta(\xi)] = Q(\xi)\ \in \mathrm{SO}(3) \quad \Rightarrow \quad Q(\xi) = \overline{Q} = \text{const.} \quad \text{and} \quad \zeta(\xi) = \overline{Q}\,\xi + \overline{b}\,, \tag{11}$$

and therefore $\mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\zeta(\xi)]] = 0$. Assuming furthermore that B is a ball of homogeneous material, we have $\zeta^{-1}(B) = B$, and the correct statement for isotropy, in our view, is then

$$\begin{aligned}\int\limits_{\xi\,\in\,B} W\Big(&\mathrm{Grad}_\xi[\varphi^\flat(\xi)]\, \overline{Q}^T,\ \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\varphi^\flat(\xi)]]\, \overline{Q}^T\, \overline{Q}^T\Big)\ \mathrm{d}\xi\\ &= \int\limits_{\xi\,\in\,B} W\Big(\mathrm{Grad}_\xi[\varphi^\flat(\xi)],\ \mathrm{GRAD}_\xi[\mathrm{Grad}_\xi[\varphi^\flat(\xi)]]\Big)\ \mathrm{d}\xi \quad \forall \overline{Q} \in \mathrm{SO}(3)\,.\end{aligned} \tag{12}$$

We denominate the latter condition as **right-global SO(3)-invariance**, which, for us, is **isotropy**. We appreciate that the right-local SO(3)-invariance condition (9) is much to restrictive in that arbitrary, inhomogeneous rotation fields are allowed instead of only constant rotations $\overline{Q}$. The reader should carefully note that we started by using a coordinate transformation $x = \zeta(\xi)$ and therefore we require in the end that $\zeta(\xi) = \overline{Q}\,\xi + \overline{b}$. There is no other coordinate transformation ζ such that $\mathrm{Grad}_\xi[\zeta(\xi)] = Q(\xi)\ \in \mathrm{SO}(3)$ everywhere, provided a minimum level of smoothness is assumed.

In the local theory the above discussion cannot distinguish between constant or non-constant rotations, since the gradient of the rotation $Q(\xi)$ is not involved. The latter might explain why one may be inclined to allow non-constant rotation fields in (9), which is forbidden for higher gradient materials [4].

References

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