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On Korn's first inequality with non-constant coefficients

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In this paper we prove a Korn-type inequality with non-constant coefficients which arises from applications in elasto-plasticity at large deformations. More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non-vanishing two-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|\Gamma} = 0\}$ and let $F_\mathrm{p}, F_\mathrm{p}^{-1} \in C^1(\bar{\Omega}, GL(3,\mathbb{R}))$ be given with $\det F_\mathrm{p}(x) \geqslant \mu^+ > 0$. Moreover, suppose that $\operatorname{Rot} F_\mathrm{p} \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists c^+ > 0 \quad \forall \phi \in H_\circ^{1,2}(\Omega,\Gamma) : \|\nabla\phi \cdot F_\mathrm{p}^{-1}(x) + F_\mathrm{p}^{-\mathrm{T}}(x) \cdot \nabla\phi^\mathrm{T}\|^2_{L^2(\Omega)} \geqslant c^+ \|\phi\|^2_{H^{1,2}(\Omega)}.$$

Clearly, this result generalizes the classical Korn's first inequality

$$\exists c^+ > 0 \quad \forall \phi \in H_\circ^{1,2}(\Omega,\Gamma) : \|\nabla\phi + \nabla\phi^\mathrm{T}\|^2_{L^2(\Omega)} \geqslant c^+ \|\phi\|^2_{H^{1,2}(\Omega)}$$

which is just our result with $F_\mathrm{p} = \mathbb{1}$. With slight modifications, we are also able to treat forms of the type

$$\|F_\mathrm{p}(x) \cdot \nabla\phi \cdot G(x) + G(x)^\mathrm{T} \cdot \nabla\phi^\mathrm{T} \cdot F_\mathrm{p}^\mathrm{T}(x)\|^p, \quad 1 < p < \infty.$$

1. Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing two-dimensional Lebesgue measure. For $a, b \in \mathbb{R}^3$, we let (a, b) denote the scalar product on $\mathbb{R}^3$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 matrices and by $\operatorname{skew}(\mathbb{M}^{3\times 3})$ the skew-symmetric real 3×3 matrices. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle A, B\rangle = \operatorname{tr}(A \cdot B^\mathrm{T})$, and subsequently we have $\|A\|^2 = \langle A, A\rangle$. With $\operatorname{Adj} A$, we denote the matrix of transposed co-factors $\operatorname{Cof}(A)$ such that $\operatorname{Adj} A = \det A \cdot A^{-1} = \operatorname{Cof}(A)^\mathrm{T}$ if $A \in GL(3,\mathbb{R})$. The identity matrix on $\mathbb{M}^{3\times 3}$ will be denoted by $\mathbb{1}$, so that $\operatorname{tr}(A) = \langle A, \mathbb{1}\rangle$. In general, we work in the context of nonlinear elasticity. For $u \in C^1(\bar{\Omega}, \mathbb{R}^3)$, we have the deformation gradient $\nabla u \in C(\bar{\Omega}, \mathbb{M}^{3\times 3})$. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega)$, $H^{1,2}(\Omega)$, $H_\circ^{1,2}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued functions. We define

$$H_\circ^{1,2}(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|\Gamma} = 0\},$$

where $\phi_{|\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω.

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2. Introduction

In the nonlinear theory of elasto-viscoplasticity at large deformation gradients, it is often assumed that the deformation gradient $F = \nabla u$ splits multiplicatively into an elastic and plastic part

$$\nabla u(x) = F(x) = F_{\mathrm{e}}(x) \cdot F_{\mathrm{p}}(x), \quad F_{\mathrm{e}}, F_{\mathrm{p}} \in GL(3,\mathbb{R}), \tag{2.1}$$

where F_{e}, F_{p} are explicitly understood to be incompatible configurations, i.e. F_{e}, $F_{\mathrm{p}} \neq \nabla \Psi$ for any $\Psi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$. In our context, we assume that this decomposition is uniquely defined up to a rigid rotation. In addition, one sometimes imposes the so-called plastic incompressibility constraint, $\det F_{\mathrm{p}}(x) = 1$. This multiplicative split, which has gained more or less permanent status in the literature, is micromechanically motivated by the kinematics of single crystals where dislocations move along fixed slip systems through the crystal lattice. The source for the incompatibility are those dislocations that did not completely transverse the crystal and consequently give rise to an inhomogeneous plastic deformation. Therefore, it seems reasonable to introduce the deviation of the plastic intermediate configuration F_{p} from compatibility as a kind of plastic *dislocation density*. This deviation should be related somehow to the quantity $\operatorname{Rot} F_{\mathrm{p}}$ and indeed later on we see the important role that is played by $\operatorname{Rot} F_{\mathrm{p}}$ (see [5,16,19,20,22,30] for more on this subject, and for applications of this theory in the engineering field look, for example, at [24,28,29]). The above split contrasts the additive decomposition into elastic and plastic parts,

$$\tfrac{1}{2}(\nabla \tilde{u} + \nabla \tilde{u}^{\mathrm{T}}) = \varepsilon(\tilde{u}(x)) = \varepsilon_{\mathrm{e}}(x) + \varepsilon_{\mathrm{p}}(x),$$

where we have set $F = 1\!\!1 + \nabla \tilde{u}$, with $\tilde{u}$ the displacement vector and where, subsequently, $\varepsilon(\tilde{u}(x))$ denotes the infinitesimal strain tensor. This decomposition is appropriate only for infinitesimal small values of $\|\nabla \tilde{u}\|$ (see, for example, [2,12,15] and the references therein). Nevertheless, the additive decomposition can be seen as a first-order approximation of (2.1).

Generally, one is then led to define an elastic energy

$$\hat{W} = \hat{W}(F_{\mathrm{e}}) = \hat{W}(\nabla u \cdot F_{\mathrm{p}}^{-1}).$$

This constitutive relation is subject to material frame indifference, i.e. must remain invariant under superimposed rigid body motions. Together with isotropy of $\hat{W}$ for $F_{\mathrm{p}} = 1\!\!1$ and the requirement that $D\hat{W}(1\!\!1) = 0$, it can be shown [6, p. 156] that there exist the so-called Lamé constants $\mu, \lambda > 0$ such that

$$\hat{W} = \hat{W}(F_{\mathrm{e}}) = \tfrac{1}{4}\mu \|F_{\mathrm{e}}^{\mathrm{T}} F_{\mathrm{e}} - 1\!\!1\|^2 + \tfrac{1}{8}\lambda \operatorname{tr}(F_{\mathrm{e}}^{\mathrm{T}} F_{\mathrm{e}} - 1\!\!1)^2 + o(\|F_{\mathrm{e}}^{\mathrm{T}} F_{\mathrm{e}} - 1\!\!1\|^2)$$

near a natural state.

2.1. No elastic rotations

In metal-plasticity one observes that the quantity $\|F_{\mathrm{e}}^{\mathrm{T}} F_{\mathrm{e}} - 1\!\!1\|$ remains pointwise small. If we incorporate this experimental fact directly into the form of the elastic energy and disregard elastic rotations, i.e. postulate in addition that $\|F_{\mathrm{e}} - 1\!\!1\|$ is small, we are led to consider elastic energies of the kind

$$\begin{aligned}
W &= W(\nabla u \cdot F_{\mathrm{p}}^{-1}) = W(F_{\mathrm{e}}) \\
&= \mu \|\tfrac{1}{2}(F_{\mathrm{e}}^{\mathrm{T}} + F_{\mathrm{e}}) - 1\!\!1\|^2 + \tfrac{1}{2}\lambda \operatorname{tr}(\tfrac{1}{2}(F_{\mathrm{e}}^{\mathrm{T}} + F_{\mathrm{e}}) - 1\!\!1)^2 \\
&= \mu \|\tfrac{1}{2}(\nabla u \cdot F_{\mathrm{p}}^{-1} + F_{\mathrm{p}}^{-\mathrm{T}} \cdot \nabla u^{\mathrm{T}}) - 1\!\!1\|^2 \\
&\qquad + \tfrac{1}{2}\lambda \operatorname{tr}(\tfrac{1}{2}(\nabla u \cdot F_{\mathrm{p}}^{-1} + F_{\mathrm{p}}^{-\mathrm{T}} \cdot \nabla u^{\mathrm{T}}) - 1\!\!1)^2,
\end{aligned}$$

where we have used that $F_{\mathrm{e}} = 1\!\!1 + (F_{\mathrm{e}} - 1\!\!1)$ and eliminated terms that are quadratic in $(F_{\mathrm{e}} - 1\!\!1)$.

If we define the corresponding functional $I : H_{\circ}^{1,2}(\Omega, \Gamma) \times C^2(\bar{\Omega}, GL(3, \mathbb{R})) \mapsto \mathbb{R}$,

$$I(u, F_{\mathrm{p}}^{-1}) := \int_{\Omega} W(\nabla u \cdot F_{\mathrm{p}}^{-1}) \, \mathrm{d}x,$$

and compute the second derivative with respect to u, we see that

$$\begin{aligned}
&D_u^2 I(u, F_{\mathrm{p}}^{-1}).(\phi, \phi) \\
&\qquad = \int_{\Omega} D^2 W(\nabla u \cdot F_{\mathrm{p}}^{-1}).(\nabla \phi, \nabla \phi) \, \mathrm{d}x \\
&\qquad = \int_{\Omega} \tfrac{1}{2}\mu \|\nabla \phi \cdot F_{\mathrm{p}}^{-1} + F_{\mathrm{p}}^{-\mathrm{T}} \cdot \nabla \phi^{\mathrm{T}}\|^2 + \tfrac{1}{4}\lambda \operatorname{tr}(\nabla \phi \cdot F_{\mathrm{p}}^{-1} + F_{\mathrm{p}}^{-\mathrm{T}} \cdot \nabla \phi^{\mathrm{T}})^2 \, \mathrm{d}x \\
&\qquad \geqslant \tfrac{1}{2}\mu \|\nabla \phi \cdot F_{\mathrm{p}}^{-1} + F_{\mathrm{p}}^{-\mathrm{T}} \cdot \nabla \phi^{\mathrm{T}}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Here, $D^2 W(\nabla u \cdot F_{\mathrm{p}}^{-1})$ is the corresponding elasticity tensor, which is not independent of the plastic evolution. Observe, however, that $D_u^2 I(u, F_{\mathrm{p}}^{-1}).(\phi, \phi)$ is independent of the deformation u itself. In the quasi-static viscoplastic setting without body forces, we then have to solve the following system of coupled partial differential and evolution equations for $u : [0, T] \times \bar{\Omega} \mapsto \mathbb{R}^3$ and $F_{\mathrm{p}} : [0, T] \times \bar{\Omega} \mapsto GL(3, \mathbb{R})$,

$$\left.\begin{aligned}
\operatorname{div} DW(\nabla u(t, x) \cdot F_{\mathrm{p}}^{-1}(t, x)) = 0, \quad x \in \Omega, \\
\frac{\mathrm{d}}{\mathrm{d}t} F_{\mathrm{p}}^{-1}(t, x) = f(\nabla u(t, x), F_{\mathrm{p}}^{-1}(t, x)), \\
u_{\Gamma}(t, x) = g(t, x), \quad x \in \Gamma, \\
F_{\mathrm{p}}^{-1}(0, x) = F_{\mathrm{p}0}^{-1},
\end{aligned}\right\} \tag{2.2}$$

with a nonlinear flow function $f : \mathbb{M}^{3 \times 3} \mapsto \mathbb{M}^{3 \times 3}$, which governs the viscoplastic evolution and is motivated by thermodynamical considerations. Here, $g(t, x)$ represents the time-dependent inhomogeneous Dirichlet boundary data and $F_{\mathrm{p}0}^{-1}$ the initial condition for the plastic evolution. This system is formally equivalent to

$$\begin{gathered}
\forall t \in [0, T] : I(u(t), F_{\mathrm{p}}^{-1}(t)) \mapsto \min, \quad u(t) \in g(t) + H_{\circ}^{1,2}(\Omega, \Gamma), \\
\frac{\mathrm{d}}{\mathrm{d}t} F_{\mathrm{p}}^{-1}(t, x) = f(\nabla u(t, x), F_{\mathrm{p}}^{-1}(t, x)), \\
F_{\mathrm{p}}^{-1}(0, x) = F_{\mathrm{p}0}^{-1}.
\end{gathered}$$

We remark that the above procedure leads to a *linear elliptic system* in u for fixed F_{p} with *non-constant coefficients*, which are determined by F_{p}, which remains valid (at least from a modelling point of view) for both large plastic deformations F_p and large deformation gradients ∇u. Note, however, that the solution u depends nonlinearly on F_p.

In the small strain case, where ε, ε_p is used, the corresponding equilibrium equations form a *linear elliptic system* in $\tilde{u}$ for fixed ε_p with *constant coefficients* and the solution depends linear on ε_p.

Our main theorem 4.10, in conjunction with the direct methods of the calculus of variations, then tells us that, for given smooth invertible F_p, the static problem

$$\left.\begin{aligned} \operatorname{div} DW(\nabla u(t,x)\cdot F_\mathrm{p}^{-1}(t,x)) = 0, &\quad x\in\Omega,\\ u_\Gamma(t,x) = g(t,x), &\quad x\in\Gamma, \end{aligned}\right\} \tag{2.3}$$

has a unique solution. We may thus dispose of (2.3) by introducing a solution operator $u = u(F_\mathrm{p}^{-1})$. The conceptual idea to treat the evolution problem is then straightforward: we write the ordinary differential equation in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\mathrm{p}^{-1}(t,x) = f(\nabla_x u(F_\mathrm{p}^{-1}), F_\mathrm{p}^{-1}(t,x)). \tag{2.4}$$

It remains to show that the right-hand side of (2.4) as a function of F_p^{-1} is locally Lipschitz in some appropriate Banach space, allowing us to apply the well-known local existence and uniqueness theorem.

2.2. The case with elastic rotations

We can adapt the above framework so as to incorporate elastic rotations. Thus we assume only that $\|F_\mathrm{e}^\mathrm{T} F_\mathrm{e} - 1\!\!1\|$ remains small. An application of the polar decomposition theorem then shows that $\|F_\mathrm{e} - R_\mathrm{e}\|$ also has to be small for a uniquely defined $R_\mathrm{e}\in O(3)$. If we repeat the above procedure with R_e instead of $1\!\!1$, we get

$$\begin{aligned} W &= W(F_\mathrm{e})\\ &= \mu\|\tfrac{1}{2}(F_\mathrm{e}^\mathrm{T}\cdot R_\mathrm{e} + R_\mathrm{e}^\mathrm{T}\cdot F_\mathrm{e}) - 1\!\!1\|^2\\ &\qquad + \tfrac{1}{2}\lambda\operatorname{tr}(\tfrac{1}{2}(F_\mathrm{e}^\mathrm{T}\cdot R_\mathrm{e} + R_\mathrm{e}^\mathrm{T}\cdot F_\mathrm{e}) - 1\!\!1)^2\\ &= \mu\|\tfrac{1}{2}(R_\mathrm{e}^\mathrm{T}\nabla u F_\mathrm{p}^{-1} + F_\mathrm{p}^{-\mathrm{T}}\nabla u^\mathrm{T} R_\mathrm{e}) - 1\!\!1\|^2\\ &\qquad + \tfrac{1}{2}\lambda\operatorname{tr}(\tfrac{1}{2}(R_\mathrm{e}^\mathrm{T}\nabla u F_\mathrm{p}^{-1} + F_\mathrm{p}^{-\mathrm{T}}\nabla u^\mathrm{T} R_\mathrm{e}) - 1\!\!1)^2, \end{aligned}$$

where we have used the fact that $F_\mathrm{e} = R_\mathrm{e} + (F_\mathrm{e} - R_\mathrm{e})$ and eliminated terms that are quadratic in $(F_\mathrm{e} - R_\mathrm{e})$. Both quantities R_e and F_p now induce inhomogeneities.

The second derivative of the corresponding functional at a given rotation R_e can be estimated by

$$D_u^2 I(u, F_\mathrm{p}^{-1}).(\phi,\phi) \geqslant \tfrac{1}{2}\mu\|R_\mathrm{e}^\mathrm{T}\cdot\nabla\phi\cdot F_\mathrm{p}^{-1} + F_\mathrm{p}^{-\mathrm{T}}\cdot\nabla\phi^\mathrm{T}\cdot R_\mathrm{e}\|^2_{L^2(\Omega)}.$$

In the presence of elastic rotations, the above system of equations (2.2) has to be complemented by either an evolution equation for R_e or some incremental device, which determines the rotation R_e uniquely at every time-step, e.g. we could set $R_\mathrm{e}^{n+1} = \operatorname{polar}(F_\mathrm{e}^n)$, where $\operatorname{polar}(F_\mathrm{e}^n)$ denotes the unique rotation associated with F_e^n by the polar decomposition theorem.

If we set out to formulate a linear problem for the deformation u, it seems impossible to use energies of the type $W = W(C, C_{\mathrm{p}})$ together with evolution equations for C_{p}. Even in the so-called physically linear setting

$$W(C, C_{\mathrm{p}}) = \langle D(x) \cdot (C - C_{\mathrm{p}}), C - C_{\mathrm{p}} \rangle,$$

where D denotes a positive-definite fourth-order tensor, and together with the assumption that $C - C_{\mathrm{p}}$ remains small, the boundary-value problem for u at given C_{p} will be highly nonlinear. This underlines again the importance of a formulation where rotations R_{e} are explicitly involved.

The fully nonlinear case, where $\hat{W} = \hat{W}(F)$ is only required to be *polyconvex*, has been investigated by the author in [25]. There one can find a local-in-time existence theorem of a suitably regularized coupled viscoplastic problem.

The theory of coercive forms has a long-dating history and we dare not trace its origins. One refers usually to [18] for a first version of Korn's inequality. By the classical Korn's first inequality, we mean

$$\exists c^+ > 0 \quad \forall \phi \in H_\circ^{1,2}(\Omega, \Gamma) : \|\nabla\phi + \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} \geqslant c^+ \|\phi\|^2_{H^{1,2}(\Omega)},$$

and we say that the classical Korn's second inequality holds if

$$\exists c^+ > 0 \quad \forall \phi \in H_\circ^{1,2}(\Omega, \Gamma) : \|\nabla\phi + \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} \geqslant c^+ \|\phi\|^2_{H^{1,2}(\Omega)}.$$

Friedrichs furnished a modern proof [9] of the above inequalities (see [4,9,13,17,27] for more on this subject). The widespread popularity of Korn's inequalities may be explained by their applicability to the linearized systems of elasticity. In this case, they yield existence, uniqueness and continuous dependence upon data. Recently, Weck [31] has shown how to circumvent Korn's second inequality in case of irregular domains and if only questions of existence are to be settled.

Ciarlet has shown [7, 8] how to extend Korn's inequalities to curvilinear coordinates, which has applications in shell theory. The main contribution of this article is to extend Korn's first inequality to non-constant coefficients that cannot be realized as metric of an underlying deformation. We rely on a theorem on coerciveness of [13], which was subsequently generalized by [4]. This theorem generalizes the Korn's second inequality to non-constant coefficients. We then proceed to show that the nullspace of our form is trivial. A compactness argument then gives the generalized Korn's first inequality. As a special case, we recover in different terms the situation of [7, p. 44].

3. Preliminaries

In the sequel, we need the following operations between $\mathbb{M}^{3\times3}$ and the Euclidean real vector space $\mathbb{R}^9$.

Definition 3.1 (Identification of $\mathbb{R}^9$ and $\mathbb{M}^{3\times3}$). We define the following operator matrix : $\mathbb{R}^9 \mapsto \mathbb{M}^{3\times3}$:

$$\operatorname{matrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Definition 3.2. We define the following operator vec : $\mathbb{M}^{3\times 3} \mapsto \mathbb{R}^9$:

$$\operatorname{vec}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{pmatrix}^{\mathrm{T}}.$$

Of course, both operations are just the canonical identifications of $\mathbb{R}^9$ and $\mathbb{M}^{3\times 3}$. We also need the following identification of $\operatorname{skew}(\mathbb{M}^{3\times 3})$ and $\mathbb{R}^3$.

Lemma 3.3. *Let $A \in \mathbb{M}^{3\times 3}$ be skew symmetric, i.e. $A = -A^{\mathrm{T}}$. If $A \neq 0$, then* $\operatorname{rank}(A) = 2$. *In addition, there is a vector $\omega \in \mathbb{R}^3$ such that*

$$A = \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix}.$$

Lemma 3.4. *Let $A \in \mathbb{M}^{3\times 3}$ be skew symmetric and let $B \in GL(3,\mathbb{R})$. If we have* $\operatorname{rank}(A \cdot B) \leqslant 1$, *then $A = 0$.*

Proof. If $\operatorname{rank}(A\cdot B) \leqslant 1$, then we can find two linear independent vectors $\tau_1, \tau_2 \in \mathbb{R}^3$ such that $(A \cdot B) \cdot \tau_1 = (A \cdot B) \cdot \tau_2 = 0$. But B is invertible and we see that $\dim(\ker(A)) \geqslant 2$, which is only possible for $A = 0$ because of lemma 3.3. □

Corollary 3.5. $\operatorname{skew}(\mathbb{M}^{3\times 3})$ *and $\mathbb{R}^3$ can be identified via*

$$\omega : \mathbb{R}^3 \mapsto \operatorname{skew}(M^{3\times 3}), \qquad \omega\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 0 & \zeta_1 & \zeta_2 \\ -\zeta_1 & 0 & \zeta_3 \\ -\zeta_2 & -\zeta_3 & 0 \end{pmatrix}$$

and ω is bijective onto its range.

Proof. Obvious. □

Definition 3.6 (Rot). We define the operator

$$\operatorname{Rot} : C^1(\bar{\Omega}, \mathbb{M}^{3\times 3}) \mapsto C(\bar{\Omega}, \mathbb{M}^{3\times 3})$$

such that we take the operator $\operatorname{rot} : C^1(\bar{\Omega}, \mathbb{R}^3) \mapsto C(\bar{\Omega}, \mathbb{R}^3)$ row-wise. For example, let $Y \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\operatorname{Rot}(Y) = \begin{pmatrix} \operatorname{rot}[Y_{11}(x,y,z), Y_{12}(x,y,z), Y_{13}(x,y,z)] \\ \operatorname{rot}[Y_{21}(x,y,z), Y_{22}(x,y,z), Y_{23}(x,y,z)] \\ \operatorname{rot}[Y_{31}(x,y,z), Y_{32}(x,y,z), Y_{33}(x,y,z)] \end{pmatrix}.$$

Lemma 3.7. *For $A \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$ with $A = -A^{\mathrm{T}}$ and $B \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$, we have*

$$\operatorname{Rot}(A \cdot B) = \operatorname{matrix}[L_B \cdot \operatorname{vec}[\nabla(\omega^{-1}(A))]] + A \cdot \operatorname{Rot}(B),$$

with a linear map $L_B : \mathbb{R}^9 \mapsto \mathbb{R}^9$,

$$L_B = \begin{pmatrix} 0 & b_{23} & -b_{22} & 0 & b_{33} & -b_{32} & 0 & 0 & 0 \\ -b_{23} & 0 & b_{21} & -b_{33} & 0 & b_{31} & 0 & 0 & 0 \\ b_{22} & -b_{21} & 0 & b_{32} & -b_{31} & 0 & 0 & 0 & 0 \\ 0 & -b_{13} & b_{12} & 0 & 0 & 0 & 0 & b_{33} & -b_{32} \\ b_{13} & 0 & -b_{11} & 0 & 0 & 0 & -b_{33} & 0 & b_{31} \\ -b_{12} & b_{11} & 0 & 0 & 0 & 0 & b_{32} & -b_{31} & 0 \\ 0 & 0 & 0 & 0 & -b_{13} & b_{12} & 0 & -b_{23} & b_{22} \\ 0 & 0 & 0 & b_{13} & 0 & -b_{11} & b_{23} & 0 & -b_{21} \\ 0 & 0 & 0 & -b_{12} & b_{11} & 0 & -b_{22} & b_{21} & 0 \end{pmatrix}.$$

Moreover, $L_B \in \mathbb{M}^{9\times 9}$ is bijective if B is bijective with

$$\det(L_B) = 2 \cdot \det(B)^3$$

and the map $B \mapsto L_B \in \mathbb{M}^{9\times 9}$ is linear.

Proof. The proof consists of simple, but long and tedious, calculations. Because this formula is the heart of the argument, we give it anyhow. First of all, we evaluate the expression $\operatorname{Rot}(A \cdot B)$ for all $A, B \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$. We write

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{pmatrix},$$

with $\bar{a}_i$, $i = 1, 2, 3$, the rows of A and

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} |b_1 & |b_2 & |b_3 \end{pmatrix},$$

with $|b_i$, $i = 1, 2, 3$, the columns of B. Then we have, of course,

$$A \cdot B = \begin{pmatrix} (\bar{a}_1, |b_1) & (\bar{a}_1, |b_2) & (\bar{a}_1, |b_3) \\ (\bar{a}_2, |b_1) & (\bar{a}_2, |b_2) & (\bar{a}_2, |b_3) \\ (\bar{a}_3, |b_1) & (\bar{a}_3, |b_2) & (\bar{a}_3, |b_3) \end{pmatrix},$$

and

$$\begin{aligned} \operatorname{Rot}(A \cdot B) &= \begin{pmatrix} \operatorname{rot}[(\bar{a}_1, |b_1) & (\bar{a}_1, |b_2) & (\bar{a}_1, |b_3)] \\ \operatorname{rot}[(\bar{a}_2, |b_1) & (\bar{a}_2, |b_2) & (\bar{a}_2, |b_3)] \\ \operatorname{rot}[(\bar{a}_3, |b_1) & (\bar{a}_3, |b_2) & (\bar{a}_3, |b_3)] \end{pmatrix} \\ &= \begin{pmatrix} \partial_y(\bar{a}_1, |b_3) - \partial_z(\bar{a}_1, |b_2) & -[\partial_x(\bar{a}_1, |b_3) - \partial_z(\bar{a}_1, |b_1)] & \partial_x(\bar{a}_1, |b_2) - \partial_y(\bar{a}_1, |b_1) \\ \partial_y(\bar{a}_2, |b_3) - \partial_z(\bar{a}_2, |b_2) & -[\partial_x(\bar{a}_2, |b_3) - \partial_z(\bar{a}_2, |b_1)] & \partial_x(\bar{a}_2, |b_2) - \partial_y(\bar{a}_2, |b_1) \\ \partial_y(\bar{a}_3, |b_3) - \partial_z(\bar{a}_3, |b_2) & -[\partial_x(\bar{a}_3, |b_3) - \partial_z(\bar{a}_3, |b_1)] & \partial_x(\bar{a}_3, |b_2) - \partial_y(\bar{a}_3, |b_1) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} (\bar a_{1_y}, |b_3) + (\bar a_1, |b_{3_y}) - (\bar a_{1_z}, |b_2) - (\bar a_1, |b_{2_z}) & 0 & 0 \\ (\bar a_{2_y}, |b_3) + (\bar a_2, |b_{3_y}) - (\bar a_{2_z}, |b_2) - (\bar a_2, |b_{2_z}) & 0 & 0 \\ (\bar a_{3_y}, |b_3) + (\bar a_3, |b_{3_y}) - (\bar a_{3_z}, |b_2) - (\bar a_3, |b_{2_z}) & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -(\bar a_{1_x}, |b_3) - (\bar a_1, |b_{3_x}) + (\bar a_{1_z}, |b_1) + (\bar a_1, |b_{1_z}) & 0 \\ 0 & -(\bar a_{2_x}, |b_3) - (\bar a_2, |b_{3_x}) + (\bar a_{2_z}, |b_1) + (\bar a_2, |b_{1_z}) & 0 \\ 0 & -(\bar a_{3_x}, |b_3) - (\bar a_3, |b_{3_x}) + (\bar a_{3_z}, |b_1) + (\bar a_3, |b_{1_z}) & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & (\bar a_{1_x}, |b_2) + (\bar a_1, |b_{2_x}) - (\bar a_{1_y}, |b_1) - (\bar a_1, |b_{1_y}) \\ 0 & 0 & (\bar a_{2_x}, |b_2) + (\bar a_2, |b_{2_x}) - (\bar a_{2_y}, |b_1) - (\bar a_2, |b_{1_y}) \\ 0 & 0 & (\bar a_{3_x}, |b_2) + (\bar a_3, |b_{2_x}) - (\bar a_{3_y}, |b_1) - (\bar a_3, |b_{1_y}) \end{pmatrix}$$

$$= \begin{pmatrix} (\bar a_{1_y}, |b_3) - (\bar a_{1_z}, |b_2) & -(\bar a_{1_x}, |b_3) + (\bar a_{1_z}, |b_1) & (\bar a_{1_x}, |b_2) - (\bar a_{1_y}, |b_1) \\ (\bar a_{2_y}, |b_3) - (\bar a_{2_z}, |b_2) & -(\bar a_{2_x}, |b_3) + (\bar a_{2_z}, |b_1) & (\bar a_{2_x}, |b_2) - (\bar a_{2_y}, |b_1) \\ (\bar a_{3_y}, |b_3) - (\bar a_{3_z}, |b_2) & -(\bar a_{3_x}, |b_3) + (\bar a_{3_z}, |b_1) & (\bar a_{3_x}, |b_2) - (\bar a_{3_y}, |b_1) \end{pmatrix}$$

$$+ \begin{pmatrix} (\bar a_1, |b_{3_y} - |b_{2_z}) & (\bar a_1, |b_{1_z} - |b_{3_x}) & (\bar a_1, |b_{2_x} - |b_{1_y}) \\ (\bar a_2, |b_{3_y} - |b_{2_z}) & (\bar a_2, |b_{1_z} - |b_{3_x}) & (\bar a_2, |b_{2_x} - |b_{1_y}) \\ (\bar a_3, |b_{3_y} - |b_{2_z}) & (\bar a_3, |b_{1_z} - |b_{3_x}) & (\bar a_3, |b_{2_x} - |b_{1_y}) \end{pmatrix}$$

$$= \begin{pmatrix} (\bar a_{1_y}, |b_3) - (\bar a_{1_z}, |b_2) & -(\bar a_{1_x}, |b_3) + (\bar a_{1_z}, |b_1) & (\bar a_{1_x}, |b_2) - (\bar a_{1_y}, |b_1) \\ (\bar a_{2_y}, |b_3) - (\bar a_{2_z}, |b_2) & -(\bar a_{2_x}, |b_3) + (\bar a_{2_z}, |b_1) & (\bar a_{2_x}, |b_2) - (\bar a_{2_y}, |b_1) \\ (\bar a_{3_y}, |b_3) - (\bar a_{3_z}, |b_2) & -(\bar a_{3_x}, |b_3) + (\bar a_{3_z}, |b_1) & (\bar a_{3_x}, |b_2) - (\bar a_{3_y}, |b_1) \end{pmatrix}$$

$$+ \begin{pmatrix} \bar a_1 \\ \bar a_2 \\ \bar a_3 \end{pmatrix} \cdot \begin{pmatrix} |b_{3_y} - |b_{2_z} & |b_{1_z} - |b_{3_x} & |b_{2_x} - |b_{1_y} \end{pmatrix}$$

$$= \begin{pmatrix} (\bar a_{1_y}, |b_3) - (\bar a_{1_z}, |b_2) & -(\bar a_{1_x}, |b_3) + (\bar a_{1_z}, |b_1) & (\bar a_{1_x}, |b_2) - (\bar a_{1_y}, |b_1) \\ (\bar a_{2_y}, |b_3) - (\bar a_{2_z}, |b_2) & -(\bar a_{2_x}, |b_3) + (\bar a_{2_z}, |b_1) & (\bar a_{2_x}, |b_2) - (\bar a_{2_y}, |b_1) \\ (\bar a_{3_y}, |b_3) - (\bar a_{3_z}, |b_2) & -(\bar a_{3_x}, |b_3) + (\bar a_{3_z}, |b_1) & (\bar a_{3_x}, |b_2) - (\bar a_{3_y}, |b_1) \end{pmatrix}$$

$$+ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{13\,y} - b_{12\,z} & b_{11\,z} - b_{13\,x} & b_{12\,x} - b_{11\,y} \\ b_{23\,y} - b_{22\,z} & b_{21\,z} - b_{23\,x} & b_{22\,x} - b_{21\,y} \\ b_{33\,y} - b_{32\,z} & b_{31\,z} - b_{33\,x} & b_{32\,x} - b_{31\,y} \end{pmatrix}$$

$$= \begin{pmatrix} (0\bar a_{1_y}, |b_3) - (1\bar a_{1_z}, |b_2) & -(2\bar a_{1_x}, |b_3) + (3\bar a_{1_z}, |b_1) \\ (6\bar a_{2_y}, |b_3) - (7\bar a_{2_z}, |b_2) & -(8\bar a_{2_x}, |b_3) + (9\bar a_{2_z}, |b_1) \\ (2\bar a_{3_y}, |b_3) - (3\bar a_{3_z}, |b_2) & -(4\bar a_{3_x}, |b_3) + (5\bar a_{3_z}, |b_1) \end{pmatrix.$$

$$\begin{matrix} (4\bar a_{1_x}, |b_2) - (5\bar a_{1_y}, |b_1) \\ (0\bar a_{2_x}, |b_2) - (1\bar a_{2_y}, |b_1) \\ (6\bar a_{3_x}, |b_2) - (7\bar a_{3_y}, |b_1) \end{matrix}\right) + A \cdot \mathrm{Rot}(B).$$

Let us now use the assumption that $A = -A^{\mathrm{T}}$ and set $\zeta = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$. We may put $A = \omega(\zeta)$. Thus

$$\nabla \zeta = \begin{pmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}.$$

This yields

$$\begin{aligned}
&\begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix} \\
&= \begin{pmatrix} ((0,\alpha_y,\beta_y), |b_3) - ((0,\alpha_z,\beta_z), |b_2) & 0 & 0 \\ ((-\alpha_y, 0, \gamma_y), b_3) - (-\alpha_z, 0, \gamma_z), |b_2) & 0 & 0 \\ ((-\beta_y, -\gamma_y, 0), |b_3) - ((-\beta_z, -\gamma_z, 0), |b_2) & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & -((0,\alpha_x,\beta_x), |b_3) + ((0,\alpha_z,\beta_z), |b_1) & 0 \\ 0 & -((-\alpha_x, 0, \gamma_x), |b_3) + ((-\alpha_z, 0, \gamma_z), |b_1) & 0 \\ 0 & -((-\beta_x, -\gamma_x, 0), |b_3) + ((-\beta_z, -\gamma_z, 0), |b_1) & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & ((0,\alpha_x,\beta_x), |b_2) - ((0,\alpha_y,\beta_y), |b_1) \\ 0 & 0 & ((-\alpha_x, 0, \gamma_x), |b_2) - ((-\alpha_y, 0, \gamma_y), |b_1) \\ 0 & 0 & ((-\beta_x, -\gamma_x, 0), |b_2) - ((-\beta_y, -\gamma_y, 0), |b_1) \end{pmatrix}.
\end{aligned}$$

Thus we arrive at

$$\begin{aligned}
&\operatorname{vec} \begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix} \\
&= \begin{pmatrix} ((0,\alpha_y,\beta_y), |b_3) - ((0,\alpha_z,\beta_z), |b_2) \\ -((0,\alpha_x,\beta_x), |b_3) + ((0,\alpha_z,\beta_z), |b_1) \\ ((0,\alpha_x,\beta_x), |b_2) - ((0,\alpha_y,\beta_y), |b_1) \\ ((-\alpha_y, 0, \gamma_y), |b_3) - (-\alpha_z, 0, \gamma_z), |b_2) \\ -((-\alpha_x, 0, \gamma_x), |b_3) + ((-\alpha_z, 0, \gamma_z), |b_1) \\ ((-\alpha_x, 0, \gamma_x), |b_2) - ((-\alpha_y, 0, \gamma_y), |b_1) \\ ((-\beta_y, -\gamma_y, 0), |b_3) - ((-\beta_z, -\gamma_z, 0), |b_2) \\ -((-\beta_x, -\gamma_x, 0), |b_3) + ((-\beta_z, -\gamma_z, 0), |b_1) \\ ((-\beta_x, -\gamma_x, 0), |b_2) - ((-\beta_y, -\gamma_y, 0), |b_1) \end{pmatrix} \\
&= \begin{pmatrix} 0 & b_{23} & -b_{22} & 0 & b_{33} & -b_{32} & 0 & 0 & 0 \\ -b_{23} & 0 & b_{21} & -b_{33} & 0 & b_{31} & 0 & 0 & 0 \\ b_{22} & -b_{21} & 0 & b_{32} & -b_{31} & 0 & 0 & 0 & 0 \\ 0 & -b_{13} & b_{12} & 0 & 0 & 0 & 0 & b_{33} & -b_{32} \\ b_{13} & 0 & -b_{11} & 0 & 0 & 0 & -b_{33} & 0 & b_{31} \\ -b_{12} & b_{11} & 0 & 0 & 0 & 0 & b_{32} & -b_{31} & 0 \\ 0 & 0 & 0 & 0 & -b_{13} & b_{12} & 0 & -b_{23} & b_{22} \\ 0 & 0 & 0 & b_{13} & 0 & -b_{11} & b_{23} & 0 & -b_{21} \\ 0 & 0 & 0 & -b_{12} & b_{11} & 0 & -b_{22} & b_{21} & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \\ \beta_x \\ \beta_y \\ \beta_z \\ \gamma_x \\ \gamma_y \\ \gamma_z \end{pmatrix} \\
&= L_B \cdot \operatorname{vec}(\nabla\zeta).
\end{aligned}$$

Therefore,

$$\operatorname{vec}(\operatorname{Rot}(A\cdot B)) = L_B\cdot \operatorname{vec}(\nabla\zeta) + \operatorname{vec}(A\cdot \operatorname{Rot} B),$$

and we get the conclusion that

$$\operatorname{Rot}(A\cdot B) = \operatorname{matrix}(L_B\cdot \operatorname{vec}(\nabla\omega^{-1}(A))) + (A\cdot \operatorname{Rot} B),$$

which is the first part of the lemma.

To find a simple direct proof of

$$\det L_B = 2\cdot(\det B)^3,$$

which shows in a few lines the above assertion, has so far eluded the efforts of the author. Instead, one has to do all the computation by hand, but I hesitate to confront the reader with them. □

LEMMA 3.8. *For $A\in C^1(\bar{\Omega},\mathbb{M}^{3\times 3})$ with $A=-A^{\mathrm{T}}$ and $B\in C^1(\bar{\Omega},\mathbb{M}^{3\times 3})$, we have*

$$\operatorname{Rot}(B\cdot A) = \hat{L}_A\cdot DB + B\cdot \operatorname{Rot}(A),$$

where, for fixed A, the map $\hat{L}_A : \mathbb{R}^{27}\mapsto \mathbb{M}^{3\times 3}$ is linear and the application $A\mapsto \hat{L}_A$ is also linear. (Here, DB denotes all partial derivatives of B with respect to (x_1,x_2,x_3).)

Proof. Is obvious from the foregoing analysis. □

Let us quickly see what happens in the standard case $B=\mathbb{1}$, which is usually involved in proving Korn's first inequality.

COROLLARY 3.9. *Assume that $A\in C^1(\bar{\Omega},\mathbb{M}^{3\times 3})$ with $A=-A^{\mathrm{T}}$. Then*

$$\operatorname{Rot}(A)=0 \quad\Rightarrow\quad A=\text{const}.$$

Proof. Retaining the same notation as in lemma 3.7, we have, for

$$A=\begin{pmatrix} 0 & \alpha & \beta\\ -\alpha & 0 & \gamma\\ -\beta & -\gamma & 0\end{pmatrix},$$

that

$$\operatorname{Rot}(A)=\begin{pmatrix} \beta_y-\alpha_z & -\beta_x & \alpha_x\\ \gamma_y & -\gamma_x-\alpha_z & \alpha_y\\ -\gamma_z & -\beta_z & -\gamma_x+\beta_y\end{pmatrix}$$

or

$$\operatorname{vec}(\operatorname{Rot}(A)) = L_{\mathbb{1}}\cdot \operatorname{vec}(\nabla\zeta).$$

Now, if $\operatorname{Rot}(A)=0$, then this implies that $\alpha_x,\alpha_y,\beta_x,\beta_z,\gamma_y,\gamma_z=0$ and

$$\left.\begin{aligned} \beta_y-\alpha_z&=0,\\ -\gamma_x-\alpha_z&=0,\\ -\gamma_x+\beta_y&=0\end{aligned}\right\}\quad\Rightarrow\quad \underbrace{\begin{pmatrix} -1 & 1 & 0\\ -1 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}}_{\text{invertible}}\cdot\begin{pmatrix}\alpha_z\\ \beta_y\\ \gamma_x\end{pmatrix}=0,$$

which yields $\alpha_z,\beta_y,\gamma_x=0$. Hence $\alpha,\beta,\gamma=\text{const}$.

This is equivalent to $A=\text{const}$.

Note that we have also implicitly shown that $L_{\mathbb{1}}:\mathbb{R}^9\mapsto\mathbb{R}^9$ is invertible. □

COROLLARY 3.10. *Assume that* $A \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$ *with* $A = -A^{\mathrm{T}}$ *and that either* $B \in GL(3, \mathbb{R})$, $B = \text{const.}$, *or* $B \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$, $B = \nabla\psi$. *Then, if* $\operatorname{Rot}(A \cdot B) = 0$, *we have* $A = \text{const.}$

Proof. From lemma 3.7, we know that $\operatorname{Rot}(A \cdot B) = 0$ implies

$$0 = \operatorname{matrix}(L_B \cdot \operatorname{vec}(\nabla\omega^{-1}(A))) + (A \cdot \operatorname{Rot} B).$$

Because B is invertible, so is L_B by way of the second part of lemma 3.7, and we can write

$$\operatorname{vec}(\nabla\omega^{-1}(A)) = L_B^{-1} \cdot \operatorname{vec}(A \cdot \operatorname{Rot} B).$$

But, in both cases, for B, we have $\operatorname{Rot} B = 0$, and if we use the assumption that $A = -A^{\mathrm{T}}$ and put

$$\zeta = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

and $A = \omega(\zeta)$, then we can write, in terms of ζ, equivalently

$$\nabla\zeta = 0$$

Hence the conclusion. □

LEMMA 3.11. *Assume that* $A \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$ *with* $A = -A^{\mathrm{T}}$ *and* $B \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$ *and that* $\operatorname{Rot}(A \cdot B) = 0$ *and* $\det B \geqslant c^+ > 0$. *Furthermore, if there is an* $x_0 \in \bar{\Omega}$ *with* $A(x_0) = 0$, *then* $A = 0$ *everywhere.*

Proof. From lemma 3.7, we know that $\operatorname{Rot}(A \cdot B) = 0$ implies

$$0 = \operatorname{matrix}(L_B \cdot \operatorname{vec}(\nabla\omega^{-1}(A))) + (A \cdot \operatorname{Rot} B).$$

Because B is invertible, so is L_B by way of the second part of lemma 3.7, and we can write

$$\operatorname{vec}(\nabla\omega^{-1}(A)) = L_B^{-1} \cdot \operatorname{vec}(A \cdot \operatorname{Rot} B).$$

Let us now use once more the assumption that $A = -A^{\mathrm{T}}$ and put

$$\zeta = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

and $A = \omega(\zeta)$. This gives, in terms of ζ, equivalently

$$\nabla\zeta = \operatorname{matrix}(L_B^{-1} \cdot \operatorname{vec}(\omega(\zeta)) \cdot \operatorname{Rot} B).$$

Consider now a smooth curve $x : [0, T] \mapsto x(t) \in \bar{\Omega}$ starting at x_0 i.e. $x(0) = x_0$. With such smooth curves we can reach every point $x \in \bar{\Omega}$. We are interested in the behaviour of ζ along these curves. We differentiate the function $t \mapsto \eta(t) := \zeta(x(t))$

to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\zeta(x(t))\\ &= \nabla\zeta(x(t))\cdot\dot{x}(t)\\ &= \mathrm{matrix}(L^{-1}_{B(x(t))}\cdot\mathrm{vec}(\omega(\zeta(x(t))))\cdot\mathrm{Rot}\,B(x(t)))\cdot\dot{x}(t)\\ &= \mathrm{matrix}(L^{-1}_{B(x(t))}\cdot\mathrm{vec}(\omega(\eta(t)))\cdot\mathrm{Rot}\,B(x(t)))\cdot\dot{x}(t).\end{aligned}$$

Together with $\eta(0)=\zeta(x(0))=\zeta(x_0)=\omega^{-1}(A(x_0))=\omega^{-1}(0)=0$, this gives the following linear system of ordinary differential equations for η along $x(t)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) &= \mathrm{matrix}(L_B(x(t))^{-1}\cdot\mathrm{vec}(\omega(\eta(t)))\cdot\mathrm{Rot}\,B(x(t)))\cdot\dot{x}(t),\\ \eta(0) &= 0.\end{aligned}$$

Because this system has a unique solution and $\eta=0$ is a solution, we must have $\zeta(x(t))$ identically 0. With the arbitrariness of $x(t)$, we see that $\zeta(x)$ is zero everywhere in $\bar{\Omega}$. But $A=\omega(\zeta)$ and we conclude $A=0$ everywhere in $\bar{\Omega}$. ☐

4. Korn-type inequalities with non-constant coefficients

LEMMA 4.1 (Ad hoc higher regularity). *Assume that* $\phi\in H^{1,2}(\Omega)$ *and* $F_{\mathrm{p}},F_{\mathrm{p}}^{-1}\in C^1(\bar{\Omega},GL(3,\mathbb{R}))$. *Furthermore, suppose that* $\mathrm{Rot}\,F_{\mathrm{p}}\in C^1(\bar{\Omega},\mathbb{M}^{3\times3})$. *If*

$$\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)+F_{\mathrm{p}}^{-\mathrm{T}}(x)\cdot\nabla\phi^{\mathrm{T}}=0,\quad x\in\Omega,$$

then $\phi\in C^2(\bar{\Omega},\mathbb{R}^3)$ *and* $A:=\nabla\phi\cdot F_{\mathrm{p}}^{-1}\in C^{1,1/2}(\bar{\Omega},\mathbb{M}^{3\times3})$.

Proof. Put $A=\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)$. Then $A=-A^{\mathrm{T}}$ and $A\in L^2(\Omega)$ because of $\phi\in H^{1,2}(\Omega)$ and $F_{\mathrm{p}}^{-1}\in C^1(\bar{\Omega},GL(3,\mathbb{R}))$. We can solve for $\nabla\phi$ because F_{p} is invertible, which gives $\nabla\phi=A\cdot F_{\mathrm{p}}$. Taking the operator Rot on both sides in the sense of distributions, we have

$$0=\mathrm{Rot}(\nabla\phi)=\mathrm{Rot}(A\cdot F_{\mathrm{p}}).$$

Now we use our formula for $\mathrm{Rot}(A\cdot F_{\mathrm{p}})$, which gives

$$0=\mathrm{matrix}[L_{F_{\mathrm{p}}}\cdot\mathrm{vec}[\nabla(\omega^{-1}(A))]]+A\cdot\mathrm{Rot}(F_{\mathrm{p}}).$$

Taking vec on both sides, we get

$$0=L_{F_{\mathrm{p}}}\cdot\mathrm{vec}[\nabla(\omega^{-1}(A))]+\mathrm{vec}(A\cdot\mathrm{Rot}(F_{\mathrm{p}})).$$

By assumption, F_{p} is everywhere invertible and so is then $L_{F_{\mathrm{p}}}$. Thus we can write this equivalently as

$$\left.\begin{aligned}\mathrm{vec}[\nabla(\omega^{-1}(A))] &= -L^{-1}_{F_{\mathrm{p}}}\cdot\mathrm{vec}(A\cdot\mathrm{Rot}(F_{\mathrm{p}})),\\ \nabla(\omega^{-1}(A)) &= -\,\mathrm{matrix}[L^{-1}_{F_{\mathrm{p}}}\cdot\mathrm{vec}(A\cdot\mathrm{Rot}(F_{\mathrm{p}}))].\end{aligned}\right\}\tag{4.1}$$

Because $A\in L^2(\Omega)$, $F_{\mathrm{p}}\in C^1(\bar{\Omega},GL(3,\mathbb{R}))$ and $\mathrm{Rot}\,F_{\mathrm{p}}\in C^1(\bar{\Omega},\mathbb{M}^{3\times3})$, we read from this formula that $\nabla(\omega^{-1}(A))\in L^2(\Omega)$. But $\nabla(\omega^{-1}(A))$ controls all first derivatives of A, which means $A\in H^{1,2}(\Omega)$. Differentiating the above expression 4.1 on both sides once more, we get that $A \in H^{2,2}(\Omega)$ since $F_{\rm p}$, $\operatorname{Rot} F_{\rm p}$ are continuously differentiable. Hence the Sobolev embedding theorem [1] yields $A \in C^{0,1/2}(\bar{\Omega}, \mathbb{M}^{3\times 3})$. Looking again at 4.1, we see that, indeed, $A \in C^{1,1/2}(\bar{\Omega}, \mathbb{M}^{3\times 3})$. Together with $\nabla\phi = A \cdot F_{\rm p}$, we see that $\nabla\phi \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$. Thus, evidently, $\phi \in C^2(\bar{\Omega}, \mathbb{R}^3)$. □

LEMMA 4.2. *Assume that $\phi \in C^1(\bar{\Omega}, \mathbb{R}^3)$ and $\phi_{|\Gamma} = 0$. Moreover, let $\Gamma \subset \partial\Omega$ be a two-dimensional smooth surface. Then there are two linear independent tangential directions, τ_1, τ_2, on Γ such that*

$$\nabla\phi(x) \cdot \tau_1(x) = 0, \qquad \nabla\phi(x) \cdot \tau_2(x) = 0.$$

Hence

$$\operatorname{rank}(\nabla\phi(x)) \leqslant 1, \quad x \in \Gamma.$$

Proof. Look at curves $s(t)$ on the surface Γ starting in $x \in \Gamma$. Then $\phi(s(t)) = 0$. Differentiating yields $\nabla\phi(s(t)) \cdot \dot{s}(t) = 0$. Because Γ is a two-dimensional smooth surface, there are two linear independent tangential directions in every point $x \in \Gamma$. If we choose the curves such that $\dot{s}(0) = \tau_{1,2}$, we see the first part of the lemma. Because then $\dim(\ker(\nabla\phi(x))) \geqslant 2$, we see the second part as well. □

THEOREM 4.3 (Trivial nullspace). *Assume that $\phi \in H^{1,2}_{\circ}(\Omega, \Gamma)$ and $F_{\rm p}, F_{\rm p}^{-1} \in C^1(\bar{\Omega}, GL(3, \mathbb{R}))$. Furthermore, suppose that* $\operatorname{Rot} F_{\rm p} \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$. *Then*

$$\|\nabla\phi \cdot F_{\rm p}^{-1}(x) + F_{\rm p}^{-\rm T}(x) \cdot \nabla\phi^{\rm T}\|^2_{L^2(\Omega)} = 0 \quad \Rightarrow \quad \phi \equiv 0.$$

Proof. Because of $\phi \in H^{1,2}_{\circ}(\Omega, \Gamma)$ and the smoothness assumptions on $F_{\rm p}$, we know by virtue of lemma 4.1 that $\phi \in C^1(\bar{\Omega}, \mathbb{R}^3)$. Therefore, we can apply lemma 4.2 to get that $\operatorname{rank}(\nabla\phi) \leqslant 1$ for $x \in \Gamma$. Now set $\nabla\phi \cdot F_{\rm p}^{-1} = A(x)$. In lemma 4.1, we showed also that $A \in C^{1,1/2}(\bar{\Omega}, \mathbb{M}^{3\times 3})$, and, of course, A is skewsymmetric. We see with lemma 3.4 that $A_{|\Gamma} = 0$. If we solve for $\nabla\phi$, we arrive at

$$\nabla\phi = A \cdot F_{\rm p}.$$

Taking now Rot on both sides in the strong sense yields $\operatorname{Rot}(A \cdot F_{\rm p}) = 0$, and we are in the position to take lemma 3.11 into account. Thus we conclude that $A = 0$ everywhere. Whence also $\nabla\phi = 0$ everywhere. From $\phi \in H^{1,2}_{\circ}(\Omega, \Gamma)$ together with Poincaré's inequality [6, p. 281], we conclude that indeed $\phi = 0$. □

Only for the convenience of the reader we give the following expression, which we need in the sequel. Let $P \in C(\bar{\Omega}, M^{3\times 3})$ and $\phi \in C^1(\bar{\Omega}, \mathbb{R}^3)$. Then, as usual,

$$\nabla\phi \cdot P = \begin{pmatrix} \dfrac{\partial\phi^1}{\partial x_1} & \dfrac{\partial\phi^1}{\partial x_2} & \dfrac{\partial\phi^1}{\partial x_3} \\ \dfrac{\partial\phi^2}{\partial x_1} & \dfrac{\partial\phi^2}{\partial x_2} & \dfrac{\partial\phi^2}{\partial x_3} \\ \dfrac{\partial\phi^3}{\partial x_1} & \dfrac{\partial\phi^3}{\partial x_2} & \dfrac{\partial\phi^3}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial\phi^1}{\partial x_1}p_{11} + \frac{\partial\phi^1}{\partial x_2}p_{21} + \frac{\partial\phi^1}{\partial x_3}p_{31} & \frac{\partial\phi^1}{\partial x_1}p_{12} + \frac{\partial\phi^1}{\partial x_2}p_{22} + \frac{\partial\phi^1}{\partial x_3}p_{32} & \frac{\partial\phi^1}{\partial x_1}p_{13} + \frac{\partial\phi^1}{\partial x_2}p_{23} + \frac{\partial\phi^1}{\partial x_3}p_{33} \\ \frac{\partial\phi^2}{\partial x_1}p_{11} + \frac{\partial\phi^2}{\partial x_2}p_{21} + \frac{\partial\phi^2}{\partial x_3}p_{31} & \frac{\partial\phi^2}{\partial x_1}p_{12} + \frac{\partial\phi^2}{\partial x_2}p_{22} + \frac{\partial\phi^2}{\partial x_3}p_{32} & \frac{\partial\phi^2}{\partial x_1}p_{13} + \frac{\partial\phi^2}{\partial x_2}p_{23} + \frac{\partial\phi^2}{\partial x_3}p_{33} \\ \frac{\partial\phi^3}{\partial x_1}p_{11} + \frac{\partial\phi^3}{\partial x_2}p_{21} + \frac{\partial\phi^3}{\partial x_3}p_{31} & \frac{\partial\phi^3}{\partial x_1}p_{12} + \frac{\partial\phi^3}{\partial x_2}p_{22} + \frac{\partial\phi^3}{\partial x_3}p_{32} & \frac{\partial\phi^3}{\partial x_1}p_{13} + \frac{\partial\phi^3}{\partial x_2}p_{23} + \frac{\partial\phi^3}{\partial x_3}p_{33} \end{pmatrix},$$

and we have, of course,

$$\nabla\phi \cdot P + P^{\mathrm{T}} \cdot \nabla\phi^{\mathrm{T}}$$

$$= \begin{pmatrix} 2\left(\frac{\partial\phi^1}{\partial x_1}p_{11} + \frac{\partial\phi^1}{\partial x_2}p_{21} + \frac{\partial\phi^1}{\partial x_3}p_{31}\right) & \frac{\partial\phi^1}{\partial x_1}p_{12} + \frac{\partial\phi^1}{\partial x_2}p_{22} + \frac{\partial\phi^1}{\partial x_3}p_{32} + \frac{\partial\phi^2}{\partial x_1}p_{11} + \frac{\partial\phi^2}{\partial x_2}p_{21} + \frac{\partial\phi^2}{\partial x_3}p_{31} & \frac{\partial\phi^1}{\partial x_1}p_{13} + \frac{\partial\phi^1}{\partial x_2}p_{23} + \frac{\partial\phi^1}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{11} + \frac{\partial\phi^3}{\partial x_2}p_{21} + \frac{\partial\phi^3}{\partial x_3}p_{31} \\ \frac{\partial\phi^1}{\partial x_1}p_{12} + \frac{\partial\phi^1}{\partial x_2}p_{22} + \frac{\partial\phi^1}{\partial x_3}p_{32} + \frac{\partial\phi^2}{\partial x_1}p_{11} + \frac{\partial\phi^2}{\partial x_2}p_{21} + \frac{\partial\phi^2}{\partial x_3}p_{31} & 2\left(\frac{\partial\phi^2}{\partial x_1}p_{12} + \frac{\partial\phi^2}{\partial x_2}p_{22} + \frac{\partial\phi^2}{\partial x_3}p_{32}\right) & \frac{\partial\phi^2}{\partial x_1}p_{13} + \frac{\partial\phi^2}{\partial x_2}p_{23} + \frac{\partial\phi^2}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{12} + \frac{\partial\phi^3}{\partial x_2}p_{22} + \frac{\partial\phi^3}{\partial x_3}p_{32} \\ \frac{\partial\phi^1}{\partial x_1}p_{13} + \frac{\partial\phi^1}{\partial x_2}p_{23} + \frac{\partial\phi^1}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{11} + \frac{\partial\phi^3}{\partial x_2}p_{21} + \frac{\partial\phi^3}{\partial x_3}p_{31} & \frac{\partial\phi^2}{\partial x_1}p_{13} + \frac{\partial\phi^2}{\partial x_2}p_{23} + \frac{\partial\phi^2}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{12} + \frac{\partial\phi^3}{\partial x_2}p_{22} + \frac{\partial\phi^3}{\partial x_3}p_{32} & 2\left(\frac{\partial\phi^3}{\partial x_1}p_{13} + \frac{\partial\phi^3}{\partial x_2}p_{23} + \frac{\partial\phi^3}{\partial x_3}p_{33}\right) \end{pmatrix}.$$

For $n = 3$ spatial dimensions, we give the following definition.

DEFINITION 4.4. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index, and let a system of operators

$$N_l : H^{1,2}(\Omega) \mapsto L^2(\Omega), \quad l = 1, \dots 9,$$

be given in such a way that for $\phi = (\phi_1, \phi_2, \phi_3) \in H^{1,2}(\Omega)$

$$N_l \cdot \phi := \sum_{s=1}^{3} \sum_{|\alpha|=1} n^l_{s\alpha}(x) \cdot D^\alpha \phi_s.$$

We say that this system is *weakly coercive* with respect to $H^{1,2}(\Omega)$ if there exists $c^+ > 0$ such that

$$\sum_{l=1}^{9} \|N_l . \phi\|^2_{2,\Omega} + \|\phi\|^2_{2,\Omega} \geqslant c^+ \|\phi\|^2_{1,2,\Omega}$$

for all $\phi \in H^{1,2}(\Omega)$.

For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3$, we define the matrix

$$N_{l_s}(x)\xi := \sum_{|\alpha|=1} n^l_{s\alpha}(x) \cdot \xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdot \xi_3^{\alpha_3}.$$

According to theorem 3.2 in [13, p. 310], we have the following result.

THEOREM 4.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $n^l_{s\alpha} \in C(\bar{\Omega}, \mathbb{R})$. Then the system N_l is weakly coercive if and only if*

$$\forall x \in \Omega : \forall \xi \in \mathbb{R}^3, \quad \xi \neq 0 \quad \Rightarrow \quad \operatorname{rank}(N_{l_s}(x)\xi) = 3,$$

$$\forall x \in \partial\Omega : \forall \xi \in \mathbb{C}^3, \quad \xi \neq 0 \quad \Rightarrow \quad \operatorname{rank}(N_{l_s}(x)\xi) = 3.$$

Proof. See [4, 13]. □

COROLLARY 4.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $P \in C(\bar{\Omega}, GL(3))$. Then the system*

$$\{N_l \phi\}_{l=1}^9 := \operatorname{vec}(\nabla\phi \cdot P + P^{\mathrm{T}} \cdot \nabla\phi^{\mathrm{T}})$$

of operators is weakly coercive over $H^{1,2}(\Omega)$.

Proof. Obviously, the coefficients of $N_l \phi$ satisfy the continuity condition of the theorem. We check the rank condition for $\xi \in \mathbb{C}^3$, $\xi \neq 0$. We have

$$\{N_l \phi\}_{l=1}^9 := \operatorname{vec}(\nabla\phi \cdot P + P^{\mathrm{T}} \cdot \nabla\phi^{\mathrm{T}}),$$

which is equal to

$$\begin{pmatrix} 2\left(\frac{\partial\phi^1}{\partial x_1}p_{11}+\frac{\partial\phi^1}{\partial x_2}p_{21}+\frac{\partial\phi^1}{\partial x_3}p_{31}\right)\\ \frac{\partial\phi^1}{\partial x_1}p_{12}+\frac{\partial\phi^1}{\partial x_2}p_{22}+\frac{\partial\phi^1}{\partial x_3}p_{32}+\frac{\partial\phi^2}{\partial x_1}p_{11}+\frac{\partial\phi^2}{\partial x_2}p_{21}+\frac{\partial\phi^2}{\partial x_3}p_{31}\\ \frac{\partial\phi^1}{\partial x_1}p_{13}+\frac{\partial\phi^1}{\partial x_2}p_{23}+\frac{\partial\phi^1}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{11}+\frac{\partial\phi^3}{\partial x_2}p_{21}+\frac{\partial\phi^3}{\partial x_3}p_{31}\\ \frac{\partial\phi^1}{\partial x_1}p_{12}+\frac{\partial\phi^1}{\partial x_2}p_{22}+\frac{\partial\phi^1}{\partial x_3}p_{32}+\frac{\partial\phi^2}{\partial x_1}p_{11}+\frac{\partial\phi^2}{\partial x_2}p_{21}+\frac{\partial\phi^2}{\partial x_3}p_{31}\\ 2\left(\frac{\partial\phi^2}{\partial x_1}p_{12}+\frac{\partial\phi^2}{\partial x_2}p_{22}+\frac{\partial\phi^2}{\partial x_3}p_{32}\right)\\ \frac{\partial\phi^2}{\partial x_1}p_{13}+\frac{\partial\phi^2}{\partial x_2}p_{23}+\frac{\partial\phi^2}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{12}+\frac{\partial\phi^3}{\partial x_2}p_{22}+\frac{\partial\phi^3}{\partial x_3}p_{32}\\ \frac{\partial\phi^1}{\partial x_1}p_{13}+\frac{\partial\phi^1}{\partial x_2}p_{23}+\frac{\partial\phi^1}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{11}+\frac{\partial\phi^3}{\partial x_2}p_{21}+\frac{\partial\phi^3}{\partial x_3}p_{31}\\ \frac{\partial\phi^2}{\partial x_1}p_{13}+\frac{\partial\phi^2}{\partial x_2}p_{23}+\frac{\partial\phi^2}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{12}+\frac{\partial\phi^3}{\partial x_2}p_{22}+\frac{\partial\phi^3}{\partial x_3}p_{32}\\ 2\left(\frac{\partial\phi^3}{\partial x_1}p_{13}+\frac{\partial\phi^3}{\partial x_2}p_{23}+\frac{\partial\phi^3}{\partial x_3}p_{33}\right) \end{pmatrix}.$$

Therefore, in this case, the matrix $N_{l_s}\xi$ looks like

$$\left(\begin{matrix} 2(\xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31}) & \xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32} & \xi_1p_{13}+\xi_2p_{23}+\xi_3p_{33}\\ 0 & \xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31} & 0\\ 0 & 0 & \xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31}\end{matrix}\right.$$

$$\begin{matrix} \xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32} & 0 & 0\\ \xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31} & 2(\xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32}) & \xi_1p_{13}+\xi_2p_{23}+\xi_3p_{33}\\ 0 & 0 & \xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32}\end{matrix}$$

$$\left.\begin{matrix} \xi_1p_{13}+\xi_2p_{23}+\xi_3p_{33} & 0 & 0\\ 0 & \xi_1p_{13}+\xi_2p_{23}+\xi_3p_{33} & 0\\ \xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31} & \xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32} & 2(\xi_1p_{13}+\xi_2p_{23}+\xi_3p_{33})\end{matrix}\right).$$

Now we show that $\operatorname{rank}(N_{l_s}) \leqslant 2$ implies $\xi = 0$, which will give the desired theorem. If $\operatorname{rank}(N_{l_s}) \leqslant 2$, then the matrices

$$E_1 := \begin{pmatrix} 2(\xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31}) & \xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32} & \xi_1p_{13}+\xi_2p_{23}+\xi_3p_{33}\\ 0 & \xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31} & 0\\ 0 & 0 & \xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31}\end{pmatrix},$$

$$E_2 := \begin{pmatrix} \xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32} & 0 & 0\\ \xi_1p_{11}+\xi_2p_{21}+\xi_3p_{31} & 2(\xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32}) & \xi_1p_{13}+\xi_2p_{23}+\xi_3p_{33}\\ 0 & 0 & \xi_1p_{12}+\xi_2p_{22}+\xi_3p_{32}\end{pmatrix},$$

$$E_3 := \begin{pmatrix} \xi_1 p_{13} + \xi_2 p_{23} + \xi_3 p_{33} & 0 & 0 \\ 0 & \xi_1 p_{13} + \xi_2 p_{23} + \xi_3 p_{33} & 0 \\ \xi_1 p_{11} + \xi_2 p_{21} + \xi_3 p_{31} & \xi_1 p_{12} + \xi_2 p_{22} + \xi_3 p_{32} & 2(\xi_1 p_{13} + \xi_2 p_{23} + \xi_3 p_{33}) \end{pmatrix}$$

must each be singular, which implies that the determinants, respectively, have to vanish. But

$$\begin{aligned} 0 &= \det E_1 = 2(\xi_1 p_{11} + \xi_2 p_{21} + \xi_3 p_{31})^3, \\ 0 &= \det E_2 = 2(\xi_1 p_{12} + \xi_2 p_{22} + \xi_3 p_{32})^3, \\ 0 &= \det E_3 = 2(\xi_1 p_{13} + \xi_2 p_{23} + \xi_3 p_{33})^3. \end{aligned}$$

This, in turn, implies that $P^{\mathrm{T}} \cdot \xi = 0$. But P is invertible and therefore $\xi = 0$. ☐

COROLLARY 4.7 (Korn's second inequality for non-constant coefficients). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $F_{\mathrm{p}}^{-1} \in C(\bar{\Omega}, GL(3))$. Then*

$$\|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)}$$

is a norm on $H^{1,2}(\Omega)$ equivalent to the standard norm.

Proof. As a consequence of weak coercivity, we get the existence of $c^+ > 0$ such that

$$\|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} \geqslant \|\phi\|^2_{H^{1,2}(\Omega)}.$$

However, the continuity of F_{p}^{-1} implies that

$$\begin{aligned} \|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} &\leqslant \|\phi\|^2_{L^2(\Omega)} + K^+ \cdot \|\nabla\phi\|^2_{L^2(\Omega)} \\ &\leqslant K^+ \|\phi\|^2_{H^{1,2}(\Omega)}. \end{aligned}$$

Hence the conclusion. ☐

REMARK 4.8. This is decisively more than Gårding's inequality, which, in the case of non-constant coefficients, together with the strict Legendre–Hadamard condition, is only valid for functions in $H^{1,2}_{\circ}(\Omega)$. Note that for constant coefficients we have more, namely coercivity over $H^{1,2}(\Omega)$ (compare with [23, p. 323]). But here we have proved a generalization of Korn's second inequality that might not have been noticed before in this special form for invertible smooth F_{p}.

For clarity of exposition, we cite the Gårding's inequality for comparison in our context.

LEMMA 4.9 (Gårding's inequality). *Let $F_{\mathrm{p}}^{-1} \in C^{0,\alpha}(\bar{\Omega}, \mathbb{M}^{3\times 3})$ be given, with*

$$\det F_{\mathrm{p}}(x) \geqslant \mu^+ > 0.$$

Then, for all $\xi, \eta \in \mathbb{R}^3$,

$$\|(\eta \otimes \xi) \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot (\eta \otimes \xi)^{\mathrm{T}}\|^2 \geqslant c^+(\mu^+)\|\eta\|^2 \cdot \|\xi\|^2,$$

and, as a consequence,

$$\exists c^+ > 0 \quad \forall \phi \in H_\circ^{1,2}(\Omega) : \|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} \geqslant c^+\|\phi\|^2_{H^{1,2}(\Omega)}.$$

Proof. See, for example, [10, p. 9]. □

We are now in a position to prove our main result.

THEOREM 4.10 (Generalized Korn's first inequality). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non-vanishing two-dimensional Lebesgue measure. Let*

$$H_\circ^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) | \phi_{|\Gamma} = 0\}$$

and let $F_{\mathrm{p}}, F_{\mathrm{p}}^{-1} \in C^1(\bar{\Omega}, GL(3,\mathbb{R}))$ be given with $\det F_{\mathrm{p}}(x) \geqslant \mu^+ > 0$. Furthermore, suppose that $\operatorname{Rot} F_{\mathrm{p}} \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists c^+ > 0 \quad \forall \phi \in H_\circ^{1,2}(\Omega, \Gamma) : \|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} \geqslant c^+\|\phi\|^2_{H^{1,2}(\Omega)}.$$

Proof. The proof proceeds now in a standard fashion by contradiction (see, for example, [6,13] for the case of the classical Korn's first inequality). Assume, on the contrary, that there is a sequence of functions $\phi_k \in H_\circ^{1,2}(\Omega, \Gamma)$ such that

$$\|\phi_k\|^2_{H^{1,2}(\Omega)} = 1, \quad \text{but } \|\nabla\phi_k \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi_k^{\mathrm{T}}\|^2_{L^2(\Omega)} \to 0.$$

Via the Rellich compact embedding of $H^{1,2}(\Omega)$ in $L^2(\Omega)$, there is a subsequence, again denoted by ϕ_k, and an element $\phi \in H_\circ^{1,2}(\Omega, \Gamma)$ with

$$\begin{aligned} \phi_k &\to \phi \quad \text{strongly in } L^2(\Omega), \\ \phi_k &\rightharpoonup \phi \quad \text{in } H^{1,2}(\Omega). \end{aligned}$$

Due to the convexity of the mapping $H \mapsto \|H \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot H^{\mathrm{T}}\|^2$, we have

$$\begin{aligned} &\|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} \\ &\qquad \leqslant \liminf_{k\to\infty} \|\nabla\phi_k \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi_k^{\mathrm{T}}\|^2_{L^2(\Omega)} = 0. \end{aligned}$$

If we apply theorem 4.3, this yields $\phi = 0$.

We show now that this subsequence is, in fact, a Cauchy sequence in the norm

$$\|\nabla u \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla u^{\mathrm{T}}\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}$$

on $H^{1,2}(\Omega)$. To see this, we note

$$\begin{aligned} &\|\nabla(\phi_k - \phi_j) \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla(\phi_k - \phi_j)^{\mathrm{T}}\|^2_{L^2(\Omega)} + \|\phi_k - \phi_j\|^2_{L^2(\Omega)} \\ &\qquad \leqslant \underbrace{\|\nabla\phi_k \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi_k^{\mathrm{T}}\|^2_{L^2(\Omega)}}_{\to 0 \text{ by assumption}} \\ &\qquad\quad + \underbrace{\|\nabla\phi_j \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi_j^{\mathrm{T}}\|^2_{L^2(\Omega)}}_{\to 0} + \underbrace{\|\phi_k - \phi_j\|^2_{L^2(\Omega)}}_{\to 0 \text{ via Rellich}}. \end{aligned}$$

Therefore, ϕ_k is also a Cauchy sequence in $H^{1,2}(\Omega)$, which means

$$\phi_k \to \phi \quad \text{strongly in } H^{1,2}(\Omega)$$

and

$$\|\phi\|^2_{H^{1,2}(\Omega)} = 1,$$

contrary to $\phi = 0$. $\square$

REMARK 4.11 (The general gradient case). The theorem shows that if $F_{\mathrm{p}} = \nabla\Psi_{\mathrm{p}}$, it is sufficient to have $F_{\mathrm{p}}, F_{\mathrm{p}}^{-1} \in C^1(\bar{\Omega}, GL(3,\mathbb{R}))$ (compare [7, p. 44]).

Interestingly enough, the above theorem can be proved using a direct argument in the gradient case $F_{\mathrm{p}} = \nabla\Psi_{\mathrm{p}}, \Psi \in C^2(\bar{\Omega}, \mathbb{R}^3)$, which mirrors the simple formula for the first Korn's inequality for functions $\phi \in H^{1,2}_{\circ}(\Omega)$.

THEOREM 4.12 (Special $H^{1,2}_{\circ}(\Omega)$ gradient case). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $F_{\mathrm{p}} = \nabla\Psi_{\mathrm{p}} \in C^1(\bar{\Omega}, \mathbb{M}^{3\times 3})$ be given with*

$$\det F_{\mathrm{p}}^{-1}(x) = \mu^+ = \text{const.} \neq 0.$$

Then

$$\exists c^+ > 0 \quad \forall \phi \in H^{1,2}_{\circ}(\Omega) : \|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)} \geqslant c^+ \|\phi\|^2_{H^{1,2}(\Omega)}.$$

Proof. For $A \in \mathbb{M}^{3\times 3}$, the Caley–Hamilton theorem tells us that

$$A^3 - \operatorname{tr}(A)\cdot A^2 + \operatorname{tr}(\operatorname{Adj} A)\cdot A - \det A \cdot \mathbb{1} = 0.$$

If $A \in GL(3,\mathbb{R})$, we can multiply this equation with A^{-1}. Taking the trace on both sides, we then have

$$\operatorname{tr}(A^2) - \operatorname{tr}(A)^2 + 2\operatorname{tr}(\operatorname{Adj} A) = 0. \tag{4.2}$$

This formula remains valid for general $A \in \mathbb{M}^{3\times 3}$. Now

$$\begin{aligned}
&\|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x)\cdot\nabla\phi^{\mathrm{T}}\|^2 \\
&\quad = 2\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)\|^2 + 2\operatorname{tr}((\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x))^2) \\
&\quad = 2\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)\|^2 - 4\operatorname{tr}(\operatorname{Adj}(\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x))) + 2\operatorname{tr}((\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)))^2 \\
&\quad \geqslant 2\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)\|^2 - 4\operatorname{tr}(\operatorname{Adj}(\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x))),
\end{aligned}$$

where use has been made of the identity 4.2. Assume that $\phi \in C_0^\infty(\Omega)$ and look at

$$\begin{aligned}
\operatorname{tr}(\operatorname{Adj}(\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x))) &= \langle \operatorname{Adj}(\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)), \mathbb{1}\rangle \\
&= \langle \operatorname{Adj}(\nabla\phi), \operatorname{Adj} F_{\mathrm{p}}^{-\mathrm{T}}(x)\rangle \\
&= \langle \operatorname{Adj}(\nabla\phi), \det F_{\mathrm{p}}^{-1}\cdot F_{\mathrm{p}}^{\mathrm{T}}\rangle \\
&= \mu^+\langle \operatorname{Adj}(\nabla\phi), F_{\mathrm{p}}^{\mathrm{T}}\rangle \\
&= \mu^+\langle \operatorname{Adj}(\nabla\phi), \nabla\Psi_{\mathrm{p}}^{\mathrm{T}}\rangle.
\end{aligned}$$

However, the Piola identity (see [6, p. 39]),

$$\operatorname{div}\operatorname{Cof}(\nabla\Psi_{\mathrm{p}}) = \operatorname{div}\operatorname{Adj}\nabla{\Psi_{\mathrm{p}}}^{\mathrm{T}} = 0,$$

together with the divergence theorem implies that ($\mu^+ = \text{const.}$)

$$\int_\Omega \lambda \langle \operatorname{Adj}(\nabla\phi), \nabla\Psi_{\rm p}^{\rm T}\rangle \,{\rm d}x = \mu^+ \int_\Omega \langle \operatorname{Adj}(\nabla\phi), \nabla\Psi_{\rm p}^{\rm T}\rangle \,{\rm d}x = 0$$

if $\phi \in C_0^\infty(\Omega)$. Therefore, upon integrating, we get

$$\begin{aligned} \|\nabla\phi \cdot F_{\rm p}^{-1}(x) + F_{\rm p}^{-\rm T}(x)\cdot\nabla\phi^{\rm T}\|^2_{L^2(\Omega)} &\geqslant 2\|\nabla\phi\cdot F_{\rm p}^{-1}(x)\|^2_{L^2(\Omega)} \\ &\geqslant 2\lambda_{\min,\bar\Omega}(F_{\rm p}^{-1}F_{\rm p}^{-\rm T})\cdot\|\nabla\phi\|^2_{L^2(\Omega)}, \end{aligned}$$

where $\lambda_{\min,\bar\Omega}(F_{\rm p}^{-1}F_{\rm p}^{-\rm T})$ denotes a lower bound for the smallest eigenvalues of $F_{\rm p}^{-1}(x)\cdot F_{\rm p}^{-\rm T}(x)$ on $\bar\Omega$. An application of Poincaré's inequality gives the result for $\phi \in C_0^\infty(\Omega)$. But $C_0^\infty(\Omega)$ is dense in $H_\circ^{1,2}(\Omega)$. □

More can be said in another special case.

THEOREM 4.13 (Special $H_\circ^{1,2}(\Omega)$ gradient case with $\Psi_{\rm p}$ a diffeomorphism). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $F_{\rm p} = \nabla\Psi_{\rm p} \in C^1(\bar\Omega, \mathbb{M}^{3\times3})$ be given with* $\det F_{\rm p}^{-1}(x) \geqslant \mu^+$ *and let $\Psi_{\rm p} : \bar\Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a C^1-diffeomorphism. Then*

$$\exists c^+ > 0 \quad \forall\phi \in H_\circ^{1,2}(\Omega) : \|\nabla\phi\cdot F_{\rm p}^{-1}(x) + F_{\rm p}^{-\rm T}(x)\cdot\nabla\phi^{\rm T}\|^2_{L^2(\Omega)} \geqslant c^+\|\phi\|^2_{H^{1,2}(\Omega)}.$$

Proof. The proof uses the fact that under the assumption that $\Psi_{\rm p} : \bar\Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a diffeomorphism, the map $x \mapsto \Psi_{\rm p}(x) =: \xi$ induces a change of variables. Indeed, if $\phi \in C_0^\infty(\Omega)$, we can uniquely define a function $\phi_{\rm e}$ by setting

$$\phi(x) = \phi_{\rm e}(\Psi_{\rm p}(x)).$$

We then get

$$\nabla\phi(x) = \nabla_\xi\phi_{\rm e}(\Psi_{\rm p}(x))\cdot\nabla_x\Psi_{\rm p}(x) \quad \text{or} \quad \nabla\phi(x)\cdot\nabla_x\Psi_{\rm p}^{-1}(x) = \nabla_\xi\phi_{\rm e}(\Psi_{\rm p}(x)).$$

For $\phi_{\rm e}$, we obtain by the simple $H_\circ^{1,2}(\Omega)$ case of Korn's first inequality that

$$\int_{\xi\in\Psi_{\rm p}(\Omega)} \|\nabla\phi_{\rm e}(\xi) + \nabla\phi_{\rm e}(\xi)^{\rm T}\|^2 \,{\rm d}\xi \geqslant 2\int_{\xi\in\Psi_{\rm p}(\Omega)} \|\nabla_\xi\phi_{\rm e}(\xi)\|^2\,{\rm d}\xi,$$

since $\phi_{\rm e}(\xi) = 0$ if $\xi \in \partial\Psi_{\rm p}(\Omega)$. Now, on applying the change of variables formula, we obtain

$$\begin{aligned} &\int_\Omega \|\nabla\phi_{\rm e}(\Psi_{\rm p}(x)) + \nabla\phi_{\rm e}(\Psi_{\rm p}(x))^{\rm T}\|^2 \det\nabla\Psi_{\rm p}(x)\,{\rm d}x \\ &\qquad\geqslant 2\int_\Omega \|\nabla_\xi\phi_{\rm e}(\Psi_{\rm p}(x))\|^2\det\nabla\Psi_{\rm p}(x)\,{\rm d}x. \end{aligned}$$

By assumption, $\det\nabla\Psi_{\rm p}(x)$ is strictly positive. Hence we can conclude that

$$\begin{aligned} &\max_\Omega(\det\nabla\Psi_{\rm p}(x))\int_\Omega \|\nabla\phi_{\rm e}(\Psi_{\rm p}(x)) + \nabla\phi_{\rm e}(\Psi_{\rm p}(x))^{\rm T}\|^2\,{\rm d}x \\ &\qquad\geqslant 2\min_\Omega(\det\nabla\Psi_{\rm p}(x))\int_\Omega \|\nabla_\xi\phi_{\rm e}(\Psi_{\rm p}(x))\|^2\,{\rm d}x. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)+F_{\mathrm{p}}^{-\mathrm{T}}(x)\cdot\nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)}\\
&\geqslant 2\frac{\min_\Omega(\det\nabla\Psi_{\mathrm{p}}(x))}{\max_\Omega(\det\nabla\Psi_{\mathrm{p}}(x))}\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)\|^2_{L^2(\Omega)}\\
&\geqslant 2\frac{\min_\Omega(\det\nabla\Psi_{\mathrm{p}}(x))}{\max_\Omega(\det\nabla\Psi_{\mathrm{p}}(x))}\mu_{\min,\bar\Omega}(F_{\mathrm{p}}^{-1}F_{\mathrm{p}}^{-\mathrm{T}})\|\nabla\phi\|^2_{L^2(\Omega)}\\
&= 2\frac{\min_\Omega(\det\nabla\Psi_{\mathrm{p}}(x)^{-1})}{\max_\Omega(\det\nabla\Psi_{\mathrm{p}}(x)^{-1})}\mu_{\min,\bar\Omega}(F_{\mathrm{p}}^{-1}F_{\mathrm{p}}^{-\mathrm{T}})\|\nabla\phi\|^2_{L^2(\Omega)}\\
&\geqslant 2\frac{\mu^+}{\max_\Omega(\det F_{\mathrm{p}}(x)^{-1})}\mu_{\min,\bar\Omega}(F_{\mathrm{p}}^{-1}F_{\mathrm{p}}^{-\mathrm{T}})\|\nabla\phi\|^2_{L^2(\Omega)}.
\end{aligned}
$$

An application of Poincaré's inequality, together with the density of $C_0^\infty(\Omega)$ in $H_\circ^{1,2}(\Omega)$, will give the result. □

For $n=2$ space dimensions, we can prove exactly the same theorem as above, but there is another theorem that might be interesting in its own right, as it can handle incompatible plastic configurations with much less regularity.

THEOREM 4.14. *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain and $F_{\mathrm{p}}\in L^\infty(\bar\Omega,\mathbb{M}^{2\times2})$ be given with* $\det F_{\mathrm{p}}^{-1}(x)=\mu^+=\text{const.}\neq 0$. *Then*

$$\exists c^+>0\quad\forall\phi\in H_\circ^{1,2}(\Omega):\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)+F_{\mathrm{p}}^{-\mathrm{T}}(x)\cdot\nabla\phi^{\mathrm{T}}\|^2_{L^2(\Omega)}\geqslant c^+\|\phi\|^2_{H^{1,2}(\Omega)}.$$

Proof. For $A\in\mathbb{M}^{2\times2}$, the Caley–Hamilton theorem tells us that

$$A^2-\mathrm{tr}(A)\cdot A-\det A\cdot\mathbb{1}=0.$$

Hence, taking the trace on both sides,

$$\mathrm{tr}(A^2)-\mathrm{tr}(A)^2=2\det A,$$

which gives, for $\phi\in C_0^\infty(\Omega)$,

$$
\begin{aligned}
\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)+F_{\mathrm{p}}^{-\mathrm{T}}(x)\cdot\nabla\phi^{\mathrm{T}}\|^2\\
&=2\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)\|^2+2\,\mathrm{tr}((\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x))^2)\\
&=2\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)\|^2+2\,\mathrm{tr}((\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)))^2-4\det(\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x))\\
&\geqslant 2\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)\|^2-4\mu^+\det(\nabla\phi).
\end{aligned}
$$

Because $\det(\nabla\phi)$ is a divergence, integrating over Ω and applying Poincaré's inequality gives the desired result, since $C_0^\infty(\Omega)$ is dense in $H_\circ^{1,2}(\Omega)$. □

5. Concluding remarks

In the case of analysing the form $\|F_{\mathrm{p}}(x)\cdot\nabla\phi+\nabla\phi^{\mathrm{T}}\cdot F_{\mathrm{p}}^{\mathrm{T}}(x)\|^2$ instead of

$$\|\nabla\phi\cdot F_{\mathrm{p}}^{-1}(x)+F_{\mathrm{p}}^{-\mathrm{T}}(x)\cdot\nabla\phi^{\mathrm{T}}\|^2,$$

we can do the same calculations as in lemma 3.11. But we see that with lemma 3.8 and invertible B, we can directly solve for $\mathrm{Rot}\,A$ and we only have to check that

$L_{\mathbb{1}} : \mathbb{R}^9 \mapsto \mathbb{R}^9$ is bijective. This can directly be seen by looking again at the computations that were done in the proof of corollary 3.9. Altogether, the whole analysis done so far carries over to this case.

The same type of coerciveness holds as well for forms of the type

$$\|G_{\mathrm{p}} \cdot \nabla\phi \cdot F_{\mathrm{p}} + F_{\mathrm{p}}^{\mathrm{T}} \cdot \nabla\phi^{\mathrm{T}} \cdot G_{\mathrm{p}}^{\mathrm{T}}\|^2,$$

with $F_{\mathrm{p}}, G_{\mathrm{p}} \in C^1(\bar{\Omega}, GL(3,\mathbb{R}))$. If we write

$$\|G_{\mathrm{p}} \cdot \nabla\phi \cdot F_{\mathrm{p}} + F_{\mathrm{p}}^{\mathrm{T}} \cdot \nabla\phi^{\mathrm{T}} \cdot G_{\mathrm{p}}^{\mathrm{T}}\|^2 = \|G_{\mathrm{p}} \cdot (\nabla\phi \cdot F_{\mathrm{p}} \cdot G_{\mathrm{p}}^{-\mathrm{T}} + G_{\mathrm{p}}^{-1} \cdot F_{\mathrm{p}}^{\mathrm{T}} \cdot \nabla\phi^{\mathrm{T}}) \cdot G_{\mathrm{p}}^{\mathrm{T}}\|^2,$$

we see immediately that we can always reduce the above case to the case

$$\|\nabla\phi \cdot C(x) + C^{\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|^2,$$

with $C \in C^1(\bar{\Omega}, GL(3,\mathbb{R}))$, since $\|G \cdot X \cdot G^{\mathrm{T}}\|$ and $\|X\|$ are equivalent norms on $\mathbb{M}^{3\times 3}$ if $G \in GL(3,\mathbb{R})$. This remark shows that we have Korn's first inequality in the case with elastic rotations as well.

A generalization of our main theorem to $L^p(\Omega)$ spaces with $1 < p < \infty$, i.e.

$$\exists c^+ > 0 \quad \forall \phi \in H_{\circ}^{1,p}(\Omega, \Gamma) \|\nabla\phi \cdot F_{\mathrm{p}}^{-1}(x) + F_{\mathrm{p}}^{-\mathrm{T}}(x) \cdot \nabla\phi^{\mathrm{T}}\|_{L^p(\Omega)}^p \geqslant c^+ \|\phi\|_{H^{1,p}(\Omega)}^p,$$

seems to be straightforward, because we get the generalization of Korn's second inequality in our situation and the $L^p(\Omega)$ setting by theorem 6 in [4, p. 530]. But, to proceed from Korn's second inequality to Korn's first inequality we did not make use of any specific $L^2(\Omega)$ property.

The question remains to be settled whether the awkward smoothness assumptions made for F_{p} and the part of the boundary Γ are sharp. Less smoothness is, of course, of utmost importance in real applications.

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