On the generalized sum of squared logarithms inequality

Waldemar Pompe* and Patrizio Neff†

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Abstract

Assume $n \geq 2$. Consider the elementary symmetric polynomials $e_k(y_1, y_2, \ldots, y_n)$ and denote by $E_0, E_1, \ldots, E_{n-1}$ the elementary symmetric polynomials in reverse order

$$E_k(y_1, y_2, \ldots, y_n) := e_{n-k}(y_1, y_2, \ldots, y_n) = \sum_{i_1 < \ldots < i_{n-k}} y_{i_1} y_{i_2} \ldots y_{i_{n-k}} \,, \quad k \in \{0, 1, \ldots, n-1\} \,.$$

Let moreover S be a nonempty subset of $\{0, 1, \ldots, n-1\}$. We investigate necessary and sufficient conditions on the function $f\colon\ I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, such that the inequality

$$f(a_1) + f(a_2) + \ldots + f(a_n) \leq f(b_1) + f(b_2) + \ldots + f(b_n) \qquad (*)$$

holds for all $a = (a_1, a_2, \ldots, a_n) \in I^n$ and $b = (b_1, b_2, \ldots, b_n) \in I^n$ satisfying

$$E_k(a) < E_k(b) \text{ for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \text{ for } k \in \{0, 1, \ldots, n-1\} \setminus S \,.$$

As a corollary, we obtain (*) if $2 \leq n \leq 4$, $f(x) = \log^2 x$ and $S = \{1, \ldots, n-1\}$, which is the sum of squared logarithms inequality previously known for $2 \leq n \leq 3$.

Key words: elementary symmetric polynomials, logarithm, matrix logarithm, inequality, characteristic polynomial, invariants, positive definite matrices, inequalities

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Contents

1 Introduction - the sum of squared logarithms inequality

In a previous contribution [1] the sum of squared logarithms inequality has been introduced and proved for the particular cases $n = 2, 3$. For $n = 3$ it reads: let $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ be given positive numbers such that

$$a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3 \,,$$
$$a_1\, a_2 + a_1\, a_3 + a_2\, a_3 \leq b_1\, b_2 + b_1\, b_3 + b_2\, b_3 \,,$$
$$a_1\, a_2\, a_3 = b_1\, b_2\, b_3 \,.$$

*Corresponding author: Waldemar Pompe, Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland, email: pompe@mimuw.edu.pl.

†Patrizio Neff, Head of Chair for Nonlinear Analysis and Modelling, Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, Thea-Leymann Str. 9, 45127 Essen, Germany, email: patrizio.neff@uni-due.de, Phone +49 201 183 4243, Fax: +49 201 183 4394

Then

$$\log^2 a_1 + \log^2 a_2 + \log^2 a_3 \le \log^2 b_1 + \log^2 b_2 + \log^2 b_3 \,.$$

The general form of this inequality can be conjectured as follows.

Definition 1.1
The standard elementary symmetric polynomials $e_1, \dots, e_{n-1}, e_n$ *are*

$$e_k(y_1, \dots, y_n) = \sum_{1 \le j_1 < j_2 < \dots < j_k \le n} y_{j_1} \cdot y_{j_2} \cdot \dots \cdot y_{j_k} \,, \quad k \in \{1, 2, \dots, n\} \,; \tag{1.1}$$

note that $e_n = y_1 \cdot y_2 \cdot \dots \cdot y_n$.

Conjecture 1.2 (Sum of squared logarithms inequality)
Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ *be given positive numbers. Then*

$$\begin{aligned} e_k(a_1, \dots, a_n) \le e_k(b_1, \dots, b_n), \quad k \in \{1, 2, \dots, n-1\}, \quad e_n(a_1, \dots, a_n) = e_n(b_1, \dots, b_n) \\ \Rightarrow \quad \sum_{i=1}^n \log^2 a_i \le \sum_{i=1}^n \log^2 b_i \,. \end{aligned} \tag{1.2}$$

Remark 1.3
Note that Conjecture 1.2 is trivial provided we have equality everywhere, i.e.

$$e_k(a_1, \dots, a_n) = e_k(b_1, \dots, b_n), \quad k \in \{1, 2, \dots, n\} \,. \tag{1.3}$$

In this case, the coefficients $a_1, \dots a_n, b_1, \dots b_n$ *are equal up to permutations, which can be seen by looking at the characteristic polynomials of two matrices with eigenvalues* $a_1, \dots, a_n$ *and* $b_1, \dots, b_n$*. From this perspective, having equality just in the last product* e_n *and strict inequality else seems to be the most difficult case.*

Based on extensive random sampling on $\mathbb{R}_+^n$ for small numbers n it has been conjectured that Conjecture 1.2 might be true for arbitrary $n \in \mathbb{N}$. The sum of squared logarithms inequality has immediate important applications in matrix analysis [7, 2] as well as in nonlinear elasticity theory [4, 5, 6, 3]. In matrix analysis it implies that the global minimizer over all rotations to

$$\inf_{Q \in \mathrm{SO}(n)} \| \operatorname{sym}_* \operatorname{Log} Q^T F \|^2 = \| \sqrt{F^T F} \|^2 \tag{1.4}$$

at given $F \in \mathrm{GL}^+(n)$ is realized by the orthogonal factor $R = \operatorname{polar}(F)$ (such that $R^T F = \sqrt{F^T F}$). Here, $\|X\|^2 := \sum_{i,j=1}^n X_{ij}^2$ denotes the Frobenius matrix norm and $\operatorname{Log} : \mathrm{GL}(n) \mapsto \mathfrak{gl}(n) = \mathbb{R}^{n \times n}$ is the multivalued matrix-logarithm, i.e. any solution $Z = \operatorname{Log} X \in \mathbb{C}^{n \times n}$ of $\exp(Z) = X$ and $\operatorname{sym}_*(Z) = \frac{1}{2}(Z^* + Z)$.

Recently, the case $n = 2$ was used to establish a polyconvexity statement in nonlinear elasticity [5, 4]. For more background information on the sum of squared logarithms inequality we refer the reader to [1].

In this paper we extend the investigation as to the validity of Conjecture 1.2 by considering arbitrary functions f instead of $f(x) = \log^2 x$. We formulate this more general problem and we are able to extend Conjecture 1.2 to the case $n = 4$. The same methods should also be useful for proving the statement for $n = 5, 6$. However, the necessary technicalities prevent us from discussing these cases in this paper.

In addition, we present ideas which might be helpful in attacking the fully general case, namely arbitrary f and arbitrary n.

2 The generalized inequality

In order to generalize Conjecture 1.2 in the directions hinted at in the introduction, we consider from now on a non-standard definition of the elementary symmetric polynomials. In fact, for

$n \geq 2$ it will be more convenient for us to reverse their numbering and define $E_0, E_1, \ldots, E_{n-1}$ by

$$E_k(y_1, \ldots y_n) := e_{n-k}(y_1, \ldots, y_n) = \sum_{i_1 < \ldots < i_{n-k}} y_{i_1} \cdot, y_{i_2} \ldots \cdot y_{i_{n-k}}\,, \quad k \in \{0, 1, \ldots, n-1\}\,. \tag{2.1}$$

In particular

$$\begin{aligned} E_0(y_1, \ldots, y_n) &:= e_n(y_1, \ldots, y_n) = y_1 \cdot y_2 \cdot \ldots \cdot y_n\,, \\ E_{n-1}(y_1, \ldots, y_n) &:= e_1(y_1, \ldots, y_n) = y_1 + y_2 + \ldots + y_n\,. \end{aligned} \tag{2.2}$$

Let $I \subset \mathbb{R}$ be an open interval and let

$$\Delta_n := \{y = (y_1, y_2, \ldots, y_n) \in I^n \,:\, y_1 \leq y_2 \leq \ldots \leq y_n\}\,. \tag{2.3}$$

Let S be a nonempty subset of $\{0, 1, \ldots, n-1\}$ and assume that $a, b \in \Delta_n$ are such that

$$E_k(a) < E_k(b) \quad \text{for } k \in S \qquad \text{and} \qquad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \ldots, n-1\} \setminus S\,. \tag{2.4}$$

In this section we investigate necessary and sufficient conditions for a (smooth) function $f\colon I \to \mathbb{R}$, such that the inequality

$$f(a_1) + f(a_2) + \ldots + f(a_n) \leq f(b_1) + f(b_2) + \ldots + f(b_n)$$

holds for all $a, b \in \Delta_n$ satisfying assumption (2.4).

Remark 2.1
The formulation of the above problem has a certain monotonicity structure: we assume that "$E(a) < E(b)$" and want to prove that "$F(a) < F(b)$". Therefore our idea is to consider a curve y connecting the points a and b, such that $E(y(t))$ "increases". Then the function $g(t) = F(y(t))$ should also increase and therefore $g'(t) > 0$ must hold. From this we are able to derive necessary and sufficient conditions on the function f.

This approach motivates the following definition.

Definition 2.2 (b *dominates* a, $a \preceq b$)
We will say that b dominates a, and denote $a \preceq b$, if there exists a piecewise differentiable mapping $y\colon [0,1] \to \Delta_n$ (i.e. y is continuous on $[0,1]$ and differentiable in all but at most countably many points) such that $y(0) = a$, $y(1) = b$, $y_i(t) \neq y_j(t)$ for all but at most countably many $t \in [0,1]$ and the functions

$$A_k(t) = E_k(y(t))\,, \qquad k \in \{0, 1, \ldots, n-1\}$$

are non-decreasing on the interval $[0,1]$.

If $a \preceq b$, then $E_k(a) = A_k(0) \leq A_k(1) = E_k(b)$, so it follows from Definition 2.2 that a, b satisfy assumption (2.4) with S being the set of all k for which $A_k(t)$ is not a constant function on $[0,1]$.

We are ready to formulate the main results of this chapter.

Theorem 2.3
Assume that $a, b \in \Delta_n$ and let $a \preceq b$. Let $S \subseteq \{0, 1, \ldots, n-1\}$ denote the set of all integers k with $E_k(a) < E_k(b)$. Moreover, assume that $f \in C^n(I)$ be such that

$$(-1)^{n+k}(x^k f'(x))^{(n-1)} \leq 0 \quad \textit{for all } x \in I \textit{ and all } k \in S\,. \tag{2.5}$$

Then the following inequality holds:

$$f(a_1) + f(a_2) + \ldots + f(a_n) \leq f(b_1) + f(b_2) + \ldots + f(b_n)\,. \tag{2.6}$$

A partially reverse statement is also true.

Theorem 2.4
Let $f \in C^n(I)$ be such that the inequality

$$f(a_1) + f(a_2) + \ldots + f(a_n) \leq f(b_1) + f(b_2) + \ldots + f(b_n) \tag{2.7}$$

holds all $a, b \in \Delta_n$ satisfying

$$E_k(a) \leq E_k(b) \quad \text{for } k \in S \qquad \text{and} \qquad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \ldots, n-1\} \setminus S \tag{2.8}$$

for some nonempty subset $S \subseteq \{0, 1, \ldots, n-1\}$. Then f satisfies property (2.5), i.e.

$$(-1)^{n+k}(x^k f'(x))^{(n-1)} \leq 0 \quad \text{for all } x \in I \text{ and all } k \in S\,. \tag{2.9}$$

In this respect, we can formulate a conjecture:

Conjecture 2.5
Let S be a nonempty subset of $\{0, 1, \ldots, n-1\}$ and assume that $a, b \in \Delta_n$ are such that

$$E_k(a) < E_k(b) \quad \text{for } k \in S \qquad \text{and} \qquad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \ldots, n-1\} \setminus S\,. \tag{2.10}$$

Then there exists a curve y satisfying the conditions from Definition 2.2 and thus $a \preceq b$.

Remark 2.6
In concrete applications of Theorem 2.3 and Theorem 2.4 one would like to know whether condition (2.4) implies $a \preceq b$. This is Conjecture 2.5. Unfortunately, we are able to prove Conjecture 2.5 only for $2 \leq n \leq 4$, $I = (0, \infty)$ and $S \subseteq \{1, 2, \ldots, n-1\}$ (see the next section).

Remark 2.7
It is easy to see that if $I = (0, \infty)$ then the function $f(x) = \log^2 x$ satisfies property (2.5) for $S = \{1, 2, \ldots, n-1\}$. Indeed, we proceed by induction on n. For $n = 2$ and $k = 1$ the property is immediate. Moreover

$$\begin{aligned}(-1)^{n+k}(x^k f'(x))^{(n-1)} &= 2(-1)^{n+k}(x^{k-1}\log x)^{(n-1)} \\ &= 2(-1)^{n+k}((k-1)x^{k-2}\log x)^{(n-2)} + 2(-1)^{n+k}(x^{k-2})^{(n-2)} \leq 0\end{aligned} \tag{2.11}$$

by the induction hypothesis, since the second summand vanishes. It remains to check property (2.5) for $k = 1$, which is also immediate.

Note also that property (2.5) is not true for $k = 0$. Therefore Theorem 2.3 and Theorem 2.4 for $f(x) = \log^2 x$ attain the following formulation:

Corollary 2.8
Assume that $a, b \in \mathbb{R}^n_+$ be such that $a \preceq b$ and $a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_n$. Then

$$\log^2(a_1) + \log^2(a_2) + \ldots + \log^2(a_n) \leq \log^2(b_1) + \log^2(b_2) + \ldots + \log^2(b_n)$$

and this inequality fails, if the constraint $a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_n$ is replaced by the weaker one $a_1 a_2 \ldots a_n \leq b_1 b_2 \ldots b_n$.

Remark 2.9
This is a weaker statement than Conjecture 1.2 since we assume that $a \preceq b$. If Conjecture 2.5 is true, then Conjecture 1.2 follows.

Remark 2.10
The function $f(x) = x^p$ $(x > 0)$ with $p \in (0, 1)$ satisfies property (2.5) for the set $S = \{0, 1, \ldots, n-1\}$. Indeed:

$$(-1)^{n+k}(x^k f'(x))^{(n-1)} = (-1)^{n+k} p(k+p-1)(k+p-2)\ldots(k+p-(n-1))x^{k+p-n}\,.$$

The above product is not greater than 0, because among the factors $k+p-1, k+p-2, \ldots, k+p-(n-1)$ there are exactly $n-1-k$ negative ones.

Similarly, the function $f(x) = x^p$ for $p \in (-1, 0)$ satisfies property (2.5) for the set $S = \{1, 2, \ldots, n-1\}$, because $p < 0$ and among the factors $k+p-1, k+p-2, \ldots, k+p-(n-1)$ there are exactly $n-k$ negative ones. On the other hand, property (2.5) is not true for $k = 0$.

Thus, similarly like above, we have

Corollary 2.11
Assume that $a, b \in (0,\infty)^n$ be such that $a \preceq b$ and $a_1a_2\dots a_n = b_1b_2\dots b_n$. If $p \in (-1,1)$, then

$$a_1^p + a_2^p + \ldots + a_n^p \le b_1^p + b_2^p + \ldots + b_n^p .$$

This inequality fails for $-1 < p < 0$ (but remains true for $0 < p < 1$) if the constraint $a_1a_2\dots a_n = b_1b_2\dots b_n$ is replaced by the weaker one $a_1a_2\dots a_n \le b_1b_2\dots b_n$.

Proof of Theorem 2.3 Let $y\colon [0,1] \to \Delta_n$ be the curve connecting points a and b like in the definition. Consider the function

$$\begin{aligned} p(t,x) &= (x+y_1(t))(x+y_2(t))\dots(x+y_n(t)) = \sum_{k=0}^{n-1} x^k E_k(y(t)) \\ &= (x+a_1)(x+a_2)\dots(x+a_n) + \sum_{k\in S} x^k A_k(t)\,, \end{aligned} \tag{2.12}$$

where $A_k(t) = E_k(y(t)) - E_k(a)$ is a non-decreasing mapping. Our goal is to show that the function

$$\eta(t) = \sum_{i=1}^{n} f(y_i(t)) \tag{2.13}$$

is non-decreasing on $[0,1]$, i.e. we show that $\eta'(t) \ge 0$ a.e. on $(0,1)$.

To this end, fix $i \in \{1,2,\dots,n\}$. Since $p(t,-y_i(t)) = 0$, we obtain

$$\tfrac{\partial}{\partial t}\, p(t,-y_i(t)) + \tfrac{\partial}{\partial x}\, p(t,-y_i(t)) \cdot (-y_i'(t)) = 0$$

for all $t \in (0,1)$ and therefore

$$\sum_{k\in S} (-y_i(t))^k A_k'(t) + \prod_{j\ne i}(y_j(t)-y_i(t)) \cdot (-y_i'(t)) = 0\,, \tag{2.14}$$

which gives

$$y_i'(t) = \sum_{k\in S} (-y_i(t))^k A_k'(t) \Big(\prod_{j\ne i}(y_j(t)-y_i(t))\Big)^{-1} .$$

From this we get

$$\begin{aligned} \eta'(t) &= \sum_{i=1}^{n} f'(y_i(t)) \cdot y_i'(t) \\ &= \sum_{i=1}^{n} f'(y_i(t)) \cdot \sum_{k\in S} (-y_i(t))^k A_k'(t) \Big(\prod_{j\ne i}(y_j(t)-y_i(t))\Big)^{-1} \\ &= \sum_{k\in S} A_k'(t) \sum_{i=1}^{n} f'(y_i(t)) \cdot (-y_i(t))^k \Big(\prod_{j\ne i}(y_j(t)-y_i(t))\Big)^{-1} . \end{aligned} \tag{2.15}$$

Fix $t \in (0,1)$ and write $y_i = y_i(t)$ for simplicity. Since $A_k'(t) \ge 0$, we will be done, if we show that

$$D := \sum_{i=1}^{n} f'(y_i) \cdot (-y_i)^k \Big(\prod_{j\ne i}(y_j - y_i)\Big)^{-1} \ge 0 \quad \text{for all } k \in S\,.$$

To this end, consider the polynomial

$$g(x) = \sum_{i=1}^{n} f'(y_i) \cdot (-y_i)^k \Big(\prod_{j\ne i}(y_j - y_i)\Big)^{-1} \cdot \prod_{j\ne i}(x - y_j)\,.$$

The degree of g equals $n-1$ and the coefficient at x^{n-1} is equal to D. Moreover,

$$g(y_i) = f'(y_i) \cdot (-y_i)^k \cdot (-1)^{n-1} \quad (i = 1,2,\dots,n)\,.$$

Therefore the function $h(x) = g(x) + (-1)^{n+k} x^k f'(x)$ has n different roots $y_1, y_2, \ldots, y_n$ in the interval I. It follows that the function

$$h^{(n-1)}(x) = (n-1)!D + (-1)^{n+k}(x^k f'(x))^{(n-1)} \tag{2.16}$$

has a root in the interval I, and since $(-1)^{n+k}(x^k f'(x))^{(n-1)} \leq 0$ for all $x \in I$, it follows that $D \geq 0$, which completes the proof of Theorem 2.3. ■

Proof of Theorem 2.4 Suppose, to the contrary, that $(-1)^{k+n}(x^k f'(x))^{(n-1)} > 0$ for some $x \in I$ and some $k \in S$. Then $(-1)^{k+n}(x^k f'(x))^{(n-1)} > 0$ holds for all x belonging to some interval J contained in I. Choose the numbers $a_1 < a_2 < \ldots < a_n$ from J and consider

$$p(t,x) = (x + a_1) \cdot (x + a_2) \cdot \ldots \cdot (x + a_n) + tx^k .$$

Then for all sufficiently small t ($0 < t < \varepsilon$), there exist different numbers $y_i(t)$ belonging to J, such that

$$p(t,x) = (x + y_1(t))(x + y_2(t)) \ldots (x + y_n(t)) .$$

Then

$$x^n + \sum_{i=0}^{n-1} E_i(a) \cdot x^i + tx^k = p(t,x) = x^n + \sum_{i=0}^{n-1} E_i(y(t)) \cdot x^i ,$$

and since $t > 0$, we see that a and $b = y(t)$ satisfy (2.8). We will be done if we show that

$$f(a_1) + f(a_2) + \ldots + f(a_n) > f(y_1(t)) + f(y_2(t)) + \ldots + f(y_n(t)) .$$

We proceed in the same way, as in the proof of Theorem 2.3. We define

$$\eta(t) = \sum_{i=1}^{n} f(y_i(t))$$

and this time we want to show that $\eta'(t) < 0$ for $0 < t < \varepsilon$.

By the Inverse Mapping Theorem (see proof of Proposition 3.4 below for a more detailed explanation), $y \in C^1(0,\varepsilon)$ and therefore

$$\eta'(t) = \sum_{i=1}^{n} f'(y_i(t)) \cdot y_i'(t) = \sum_{i=1}^{n} f'(y_i(t)) \cdot (-y_i(t))^k \Big(\prod_{j \neq i} (y_j(t) - y_i(t)) \Big)^{-1} . \tag{2.17}$$

Now, like previously, write $y_i = y_i(t)$ for simplicity. Our goal is therefore to prove that

$$D := \sum_{i=1}^{n} f'(y_i) \cdot (-y_i)^k \Big(\prod_{j \neq i} (y_j - y_i) \Big)^{-1} < 0 .$$

Consider the polynomial

$$g(x) = \sum_{i=1}^{n} f'(y_i) \cdot (-y_i)^k \Big(\prod_{j \neq i} (y_j - y_i) \Big)^{-1} \cdot \prod_{j \neq i} (x - y_j) .$$

The degree of g equals $n-1$ and the coefficient at x^{n-1} is equal to D. Moreover, the function $h(x) = g(x) + (-1)^{n+k} x^k f'(x)$ has n different roots $y_1, y_2, \ldots, y_n$ in the interval J. It follows that the function

$$h^{(n-1)}(x) = (n-1)!D + (-1)^{n+k}(x^k f'(x))^{(n-1)}$$

has a root in the interval J. And since $(-1)^{n+k}(x^k f'(x))^{(n-1)} > 0$ for all $x \in J$, it follows that $D < 0$, which completes the proof of Theorem 2.4. ■

3 Construction of the connecting curve

In this section we prove that condition (2.4) implies $a \preceq b$, if $2 \leq n \leq 4$, $I = (0, \infty)$ and $S \subseteq \{1, 2, \ldots, n{-}1\}$. However, we start with a construction of the desired curve for a general interval I, integer $n \geq 2$ and set $S \subseteq \{0, 1, \ldots, n{-}1\}$.

For $a, b \in \Delta_n$, we say that $a < b$, if $a \neq b$ and $E_k(a) \leq E_k(b)$ for all $k = 0, 1, \ldots, n{-}1$. We say that $a \leq b$, if $a < b$ or $a = b$.

Definition 3.1
For $a < b$ denote by $\mathcal{C}(a, b)$ the set of all piecewise differentiable (i.e. continuous and differentiable in all but at most countably many points) curves y in Δ_n satisfying:

(a) the curve $y(t)$ starts at a (i.e. $y(0) = a$, if the curve $y(t)$ is parametrized by the interval $[0, \varepsilon]$);

(b) $y(t) \in \operatorname{int}(\Delta_n)$ for all but at most countable many values t;

(c) the mappings $E_k(y(t))$ are non-decreasing on t and $E_k(y(t)) \leq E_k(b)$ for all t and each $k = 0, 1, \ldots, n{-}1$.

Note that a curve in $\mathcal{C}(A, b)$ does not necessarily end at the point b.

Proposition 3.2
Let $n \geq 2$ be a positive integer and let S be a nonempty subset of $\{0, 1, \ldots, n{-}1\}$. Let moreover $a, b \in \Delta_n$ be such that (2.4) holds. Furthermore, suppose that for all $c \in \Delta_n$ with $a \leq c < b$ the set $\mathcal{C}(c, b)$ is nonempty. Then $a \preceq b$.

Proof. Each element (curve) of $\mathcal{C}(a, b)$ is a (closed) subset of Δ_n. We equip the set $\mathcal{C}(a, b)$ with the inclusion relation $\subseteq$, obtaining a nonempty partially ordered set $(\mathcal{C}(a, b), \subseteq)$. We are going to show that each chain $\{y_i\}_{i \in \mathcal{I}}$ has an upper bound in $\mathcal{C}(a, b)$.

To achieve this, consider the curve

$$y_0 = \overline{\bigcup_{i \in \mathcal{I}} y_i}\,.$$

Then obviously y_0 satisfies conditions (a) and (c) of Definition 3.1. To prove (b) assume that y_0 is parametrized on $[0, 1]$. Then for each positive integer k the curve y_k, defined as the restriction of y_0 to the interval $[0, 1 - \frac{1}{k}]$, is contained in some curve $y_i \in \mathcal{C}(a, b)$ of the given chain $\{y_i\}$. Therefore $y_k(t)$ is piecewise differentiable and satisfies condition (b) for each positive integer k. Moreover,

$$y_0 = \overline{\bigcup_{k=1}^{\infty} y_k}\,.$$

Hence y_0 is piecewise differentiable and satisfies (b) as well.

Now, by the Kuratowski-Zorn lemma, there exists a maximal element y in $(\mathcal{C}(a, b), \subseteq)$. We show that y is a desired curve connecting the points a and b, which will imply that $a \preceq b$.

To this end, it is enough to show that, if the curve y is parametrized on $[0, 1]$, then $y(1) = b$. Suppose, to the contrary, that $y(1) = c \neq b$. Then $a \leq c < b$, and hence the set $\mathcal{C}(c, b)$ is nonempty. Thus the curve y can be extended beyond the point c, which contradicts the fact that y is a maximal element in $\mathcal{C}(a, b)$. This completes the proof of Proposition 3.2. ∎

From now on assume that $I = (0, \infty)$ and S is a nonempty subset of $\{1, 2, \ldots, n{-}1\}$.

In order to prove that (2.4) implies $a \preceq b$, it suffices to show that the sets $\mathcal{C}(a, b)$ for $a, b \in \Delta_n$ with $a < b$ are nonempty. This is implied by the following conjecture, which we will prove later for $n \leq 4$.

Conjecture 3.3
Let $n \geq 2$ be an integer and $a \in \Delta_n$. Let S be a nonempty subset of $\{1, 2, \ldots, n{-}1\}$ with the property that there exist $A_k > 0$ for $k \in S$ such that all the roots of the polynomial

$$q(x) = (x + a_1)(x + a_2) \ldots (x + a_n) + \sum_{k \in S} A_k x^k$$

are real (and hence negative). Then there exist continuous on $[0, \varepsilon]$, differentiable on $(0, \varepsilon)$ and nondecreasing mappings $B_k : [0, \varepsilon] \to \mathbb{R}$ ($k \in S$) with $B_k(0) = 0$ such that $\sum_{k \in S} B_k(t)$ is *increasing on* $[0,\varepsilon]$ *and for all sufficiently small values of* $t>0$ *the polynomial*

$$(x+a_1)(x+a_2)\dots(x+a_n)+\sum_{k\in S}B_k(t)x^k$$

has n *distinct real (and hence negative) roots.*

Now we show how Conjecture 3.3 implies that the sets $\mathcal{C}(a,b)$ are nonempty.

Proposition 3.4
Let n *and* S *be such that the conjecture holds. Let moreover* $a,b\in\Delta_n$ *be such that (2.4) holds. Then the set* $\mathcal{C}(a,b)$ *is nonempty.*

Proof. Consider the polynomials

$$p(x)=(x+a_1)(x+a_2)\dots(x+a_n)\quad\text{and}\quad q(x)=(x+b_1)(x+b_2)\dots(x+b_n)\,.$$

Then

$$q(x)-p(x)=\sum_{k=0}^{n-1}(E_k(b)-E_k(a))x^k=\sum_{k\in S}A_kx^k\,,$$

where $A_k>0$ for all $k\in S$. According to the conjecture, there exist continuous on $[0,\varepsilon]$ and differentiable on $(0,\varepsilon)$ nondecreasing mappings $B_k:[0,\varepsilon]\to\mathbb{R}$, with $B_k(0)=0$ such that $\sum_{k\in S}B_k(t)$ is increasing on $[0,\varepsilon]$ and for all $t\in(0,\varepsilon)$ the polynomial

$$p(x)+\sum_{k\in S}B_k(t)x^k$$

has n distinct real (and hence negative) roots $-y_n(t)<-y_{n-1}(t)<\ldots<-y_1(t)<0$. We show that $y(t)=(y_1(t),y_2(t),\dots,y_n(t))$ defines a differentiable curve (parametrized on $[0,\varepsilon]$) that belongs to $\mathcal{C}(a,b)$, provided ε is chosen in such a way that $B_k(\varepsilon)\le A_k$ for $k\in S$.

Consider the following mapping $\Psi\colon\overline{\Delta_n}\to\Psi(\overline{\Delta_n})$ given by

$$\Psi(y)=(E_{n-1}(y),E_{n-2}(y),\dots,E_0(y))\,.$$

Then it follows from Remark 1.3 that the mapping Ψ is injective, hence Ψ is a continuous bijection defined on a closed subset of $\mathbb{R}^n$. Therefore the mapping Ψ^{-1} is continuous and thus

$$y(t)=\Psi^{-1}(a+(B_0(t),B_1(t),\dots,B_{n-1}(t)))\quad(t\in[0,\varepsilon])$$

(here we put $B_k(t)=0$ for $k\notin S$) is a curve starting at a. Moreover $y(t)\in\Delta_n$. Hence condition (a) is satisfied. Since $y(t)\in\operatorname{int}(\Delta_n)$ for all $t\in(0,\varepsilon)$, condition (b) holds. It is also clear that (c) is satisfied, since $E_k(y(t))=E_k(a)+B_k(t)\le E_k(a)+A_k=E_k(b)$ for all $k\in\{0,1,\dots,n-1\}$.

So it remains to prove that $y(t)$ is differentiable on $(0,\varepsilon)$. This however is a consequence of the Inverse Mapping Theorem, if we show that

$$\det[D\Psi(y)]\neq0\quad\text{for all }y\in\operatorname{int}(\Delta_n).$$

To this end, let $V(y)$ be the $n\times n$ Vandermonde-type matrix given by $V_{ij}(y)=(-y_i)^{n-j}$ $(1\le i,j\le n)$. This matrix is obtained from the standard Vandermonde matrix

$$W(-y_1,-y_2,\dots,-y_n)=\begin{pmatrix}1&-y_1&(-y_1)^2&\cdots&(-y_1)^{n-1}\\1&-y_2&(-y_2)^2&\cdots&(-y_2)^{n-1}\\1&-y_3&(-y_3)^2&\cdots&(-y_3)^{n-1}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&-y_n&(-y_n)^2&\cdots&(-y_n)^{n-1}\end{pmatrix}\tag{3.1}$$

by reversing the order of columns of W.

Then by the formula

$$t^{n-1}+\sum_{k=0}^{n-2}t^kE_k(z_1,z_2,\dots,z_{n-1})=(t+z_1)(t+z_2)\dots(t+z_{n-1})\,,\tag{3.2}$$

we infer that

$$V(y) \cdot D\Psi(y) = \operatorname{diag}\Big(\prod_{j\neq 1}(y_j - y_1), \prod_{j\neq 2}(y_j - y_2), \dots, \prod_{j\neq n}(y_j - y_n)\Big). \tag{3.3}$$

It is well-known that

$$\det[V(y)] = \prod_{i<j}(y_j - y_i) \neq 0 \quad (y \in \operatorname{int}\Delta_n).$$

Therefore we obtain

$$\det[D\Psi(y)] = \prod_{i<j}(y_i - y_j) \neq 0 \quad (y \in \operatorname{int}\Delta_n),$$

which completes the proof of Proposition 3.4. ■

Lemma 3.5
Assume that $n \geq 3$ is odd and let $0 < a_1 \leq a_2 \leq \ldots \leq a_n$. Let moreover $A_k \geq 0$ for $k = 1, 2, \dots, (n-1)/2$ with at least one A_k not equal to 0. Consider the polynomials

$$P(x) = (x + a_1)(x + a_2)\dots(x + a_n) + \sum_{k=1}^{(n-1)/2} A_k x^{2k-1},$$

$$Q(x) = (x + a_1)(x + a_2)\dots(x + a_n) + \sum_{k=1}^{(n-1)/2} A_k x^{2k}. \tag{3.4}$$

Then the polynomial P has exactly one root in the interval $(-a_1, 0)$ and at most two roots in the interval $(-a_n, -a_{n-1})$. Moreover, the polynomial Q has exactly one root in the interval $(-\infty, -a_n)$ and at most two roots in the interval $(-a_2, -a_1)$.

Proof. That P has exactly one root in $(-a_1, 0)$ follows immediately from the observation that $P(-a_1) < 0$, $P(0) > 0$ and $P'(x) > 0$ on $(-a_1, 0)$.

Now we show that Q has exactly one root in $(-\infty, -a_n)$.

Dividing the equation $Q(x) = 0$ by $x^n a_1 a_2 \dots a_n$ and substituting $z = 1/x$ and $b_i = 1/a_i$, yields the equation $P_0(z) = 0$, where

$$P_0(z) = (z + b_1)(z + b_2)\dots(z + b_n) + \sum_{k=1}^{(n-1)/2} B_k z^{2k-1}$$

for some nonnegative numbers B_k, not all equal to 0. We already know that P_0 has exactly one root in the interval $(-b_n, 0)$, so it follows that Q has exactly one root in the interval $(-\infty, -a_n)$.

Now we prove that Q has at most two roots in the interval $(-a_2, -a_1)$. To the contrary, suppose that Q has at least 3 roots in $(-a_2, -a_1)$. Since $Q(-a_2) > 0$ and $Q(-a_1) > 0$, it follows that Q has an even number, and hence at least four, roots in the interval $(-a_2, -a_1)$.

Let $0 > -c_1 \geq -c_2 \geq \ldots \geq -c_{n-1}$ be the roots of $p'(x) = 0$, where

$$p(x) = (x + a_1)(x + a_2)\dots(x + a_n). \tag{3.5}$$

Then $a_1 < c_1 < a_2$. The polynomial $Q(x)$ is decreasing on the interval $[-a_2, -c_1]$, so it has at most one root in this interval. Therefore the polynomial Q has at least three roots in the interval $(-c_1, -a_1)$, and consequently the equation $Q''(x) = 0$ has a root in $(-c_1, -a_1)$. But $Q''(x) > 0$ for all $x > -c_1$, a contradiction. Hence Q must have at most two roots in $(-a_2, -a_1)$.

Finally, to prove that P has at most two roots in the interval $(-a_n, -a_{n-1})$, divide the equation $P(x) = 0$ by $x^n a_1 a_2 \dots a_n$ and substitute $z = 1/x$ and $b_i = 1/a_i$. This reduces to the equation $Q_0(z) = 0$, where

$$Q_0(z) = (z + b_1)(z + b_2)\dots(z + b_n) + \sum_{k=1}^{(n-1)/2} B_k z^{2k}$$

for some nonnegative numbers B_k, not all equal to 0. We already know that Q_0 has at most two roots in the interval $(-b_{n-1}, -b_n)$, so it follows that P has at most two roots in the interval $(-a_n, -a_{n-1})$. This completes the proof of Lemma 3.5. ■

The same proof yields an analogous result for even values of n.

Lemma 3.6
Assume that $n \geq 2$ is even and let $0 < a_1 \leq a_2 \leq \ldots \leq a_n$. Let moreover $A_k \geq 0$ for $k = 1, 2, \ldots, n/2$ and not all of the A_k's are equal to 0. Consider the polynomials

$$P(x) = (x + a_1)(x + a_2)\ldots(x + a_n) + \sum_{k=1}^{n/2} A_k x^{2k-1}\,,$$
$$Q(x) = (x + a_1)(x + a_2)\ldots(x + a_n) + \sum_{k=1}^{n/2-1} A_k x^{2k}\,. \tag{3.6}$$

Then the polynomial P has exactly one root in each of the intervals $(-\infty, -a_n)$ and $(-a_1, 0)$ and Q has at most two roots in each of the intervals $(-a_n, -a_{n-1})$ and $(-a_2, -a_1)$.

Proof. The same proof as that for Lemma 3.5 can be used. ∎

Now we turn to the proof of Conjecture 3.3 for $2 \leq n \leq 4$ and an arbitrary nonempty set $S \subseteq \{1, 2, \ldots, n-1\}$.

We first make some useful general remarks.

Let $I(a) = \{i \in \{1, 2, \ldots, n-1\} \;:\; a_i = a_{i+1}\}$. If $I(a)$ is empty, then the conjecture holds. Indeed, if $k \in S$, then all the roots of the polynomial

$$(x + a_1)(x + a_2)\ldots(x + a_k) + tx^k$$

are, for all sufficiently small $t > 0$, real and distinct.

On the other hand, if $I(a) = \{1, 2, \ldots, n-1\}$, then only the set $S = \{1, 2, \ldots, n-1\}$ satisfies the assumptions of the conjecture. Indeed, suppose that $l \notin S$ and let $-b_1 \geq -b_2 \geq \ldots \geq -b_n$ be the roots of

$$q(x) = (x + a_1)^n + \sum_{k \in S} A_k x^k\,.$$

Then by the inequality of arithmetic and geometric means, we obtain

$$\frac{E_l(a)}{\binom{n}{l}} = \frac{E_l(b)}{\binom{n}{l}} \geq (E_0(b))^{(n-l)/n} = (E_0(a))^{(n-l)/n} = \frac{E_l(a)}{\binom{n}{l}}\,, \tag{3.7}$$

and hence $b_1 = b_2 = \ldots = b_n$. Since $E_0(a) = E_0(b)$, it follows that $a = b$, i.e. $A_k = 0$ for all $k \in S$. A contradiction.

Let I be a non-empty subset of $\{1, 2, \ldots, n-1\}$. We observe that the conjecture is true for a set S and all $a \in \Delta_n$ with $I(a) = I$, if it is true for a set $T = \{n-k \;:\; k \in S\}$ and all $b \in \Delta_n$ with $I(b) = \{n-i \;:\; i \in I\}$. Indeed: if all the roots of the polynomial

$$q(x) = (x + a_1)(x + a_2)\ldots(x + a_n) + \sum_{k \in S} A_k x^k$$

are real, then substituting $x = 1/z$ and $a_i = 1/b_i$, we infer that all the roots of the polynomial

$$r(z) = (z + b_1)(z + b_2)\ldots(z + b_n) + \sum_{l \in T} B_l z^l$$

are real. Hence there exist continuous on $[0, \varepsilon]$, differentiable on $(0, \varepsilon)$ and nondecreasing mappings $C_l(t)$ with $C_l(0) = 0$ such that the polynomial

$$(z + b_1)(z + b_2)\ldots(z + b_n) + \sum_{l \in T} C_l(t) z^l$$

has n distinct real roots. Substituting $z = 1/x$ and $b_i = 1/a_i$, we infer that the polynomial

$$(x + a_1)(x + a_2)\ldots(x + a_n) + \sum_{k \in S} C_{n-k}(t) x^k$$

has n distinct real roots.

For $n = 2$ the only possibility for the set S is $\{1\}$ and it is enough to notice that the polynomial $(x + a_1)(x + a_2) + tx$ has two distinct real roots for any $t > 0$.

Assume now $n = 3$. Then, in view of the above remarks, we have to consider two cases: 1) $a_1 < a_2 = a_3$; 2) $a_1 = a_2 = a_3$.

1) If $2 \notin S$, then the condition of Conjecture 3.3 can not be satisfied since, according Lemma 3.5, the polynomial

$$P(x) = (x + a_1)(x + a_2)^2 + A_1 x$$

has only one real root for all $A_1 > 0$. We can therefore assume $2 \in S$, and for all sufficiently small $t > 0$, the polynomial

$$(x + a_1)(x + a_2)^2 + tx^2$$

has three distinct real roots.

2) According to the above remarks, $S = \{1, 2\}$. Then the polynomial $(x + a_1)^3 + ta_1 x + tx^2$ has 3 distinct real roots for all sufficiently small $t > 0$.

Assume $n = 4$. In this case we have 5 possibilities: 1) $a_1 = a_2 < a_3 < a_4$; 2) $a_1 < a_2 = a_3 < a_4$; 3) $a_1 < a_2 = a_3 = a_4$; 4) $a_1 = a_2 < a_3 = a_4$; 5) $a_1 = a_2 = a_3 = a_4$.

1) We note that $S \neq \{2\}$, since, by Lemma 3.6, the polynomial

$$(x + a_1)^2(x + a_3)(x + a_4) + A_2 x^2 \quad \text{for } A_2 > 0$$

has at most two real roots. Therefore S contains an odd integer k. Then for all sufficiently small $t > 0$, the polynomial $(x + a_1)^2(x + a_3)(x + a_4) + tx^k$ has four distinct real roots.

2) Note that $2 \in S$, since otherwise, by Lemma 2, the polynomial

$$(x + a_1)(x + a_2)^2(x + a_4) + A_1 x + A_3 x^3 \quad \text{for } A_1, A_3 > 0$$

has at most two real roots. Then for all sufficiently small $t > 0$, the polynomial

$$(x + a_1)(x + a_2)^2(x + a_4) + tx^2$$

has four distinct real roots.

3) We observe that $\{1, 2\} \subset S$ or $\{2, 3\} \subset S$, since otherwise, by Lemma 2, each of the polynomials

$$(x + a_1)(x + a_2)^3 + A_1 x + A_3 x^3 \quad \text{and} \quad (x + a_1)(x + a_2)^3 + A_2 x^2 \quad \text{for } A_1, A_2, A_3 > 0$$

has at most two real roots. Moreover, we prove that $S \neq \{1, 2\}$.

Suppose that the polynomial $(x + a_1)(x + a_2)^3 + A_1 x + A_2 x^2$ has four real roots. Let $Q_1(x) = (x+a_1)(x+a_2)^3$ and $Q_2(x) = A_1 x + A_2 x^2$. Let $-c \neq a_2$ be the root of the polynomial $Q_1'(x)$ and let $-d$ be the root of $Q_2'(x)$.

If $d < c$, then Q is decreasing on $(-\infty, -c]$, so Q has at most one root in this interval. Therefore Q has at least 3 roots in the interval $(-c, 0)$. Thus $Q''(x)$ has a root in the interval $(-c, 0)$, which is impossible, since $Q''(x) > 0$ on $(-c, 0)$.

If $a_2 \geq d \geq c$, then Q is increasing on the intervals $[-c, 0)$ and $(-\infty, -d]$, so Q must have at least two roots in the interval $(-d, -c)$. But $Q(x) < 0$ on this interval.

Finally, if $d > a_2$, then Q may only have roots in the union $(-\infty, a_2) \cup (-a_1, 0)$. But Q is increasing on $(-a_1, 0)$, so Q has 3 roots in $(-\infty, a_2)$. This however is impossible, since $Q''(x) > 0$ for $x \in (-\infty, a_2)$. Thus $\{2, 3\} \subseteq S$ and the polynomial

$$(x + a_1)(x + a_2)^3 + tx^2(x + a_2)$$

has for all sufficiently small $t > 0$ four distinct roots.

4) Since the polynomial $(x + a_1)^2(x + a_3)^2 + A_2 x^2$ has no real roots, $1 \in S$ or $3 \in S$. Then the polynomial $(x + a_1)^2(x + a_3)^2 + tx^k$ for $k = 1, 3$ has for all sufficiently small $t > 0$ four distinct real roots.

5) In view of the above remarks, $S = \{1, 2, 3\}$. Consider

$$r(x) = (x + a_1)^4 + tx^3 + 2ta_1 x^2 + t(a_1^2 - t^2)x = (x + a_1)^4 + tx((x + a_1)^2 - t^2)\,.$$

Then for all sufficiently small $t > 0$, $a_1^2 - t^2 > 0$, and the polynomial r has four distinct real roots, because

$$r(-a_1 - 2t) = t^3(10t - 3a_1) < 0\,,\ r(-a_1) = a_1 t^3 > 0 \text{ and } r(-a_1 + 2t) = t^3(22t - 3a_1) < 0\,.$$

Thus we have proved:

Corollary 3.7
Conjecture 3.3 is true if $2 \le n \le 4$ *and* S *is an arbitrary subset of* $\{1, 2, \ldots, n-1\}$.

This implies that the sum of squared logarithms inequality (Conjecture 1.2) holds also for $n = 4$.

Corollary 3.8 (Sum of squared logarithms inequality for $n = 4$)
Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 > 0$ *be given positive numbers such that*

$$\begin{gathered} a_1 + a_2 + a_3 + a_4 \le b_1 + b_2 + b_3 + b_4 \,, \\ a_1\, a_2 + a_1\, a_3 + a_2\, a_3 + a_1\, a_4 + a_2\, a_4 + a_3\, a_4 \le b_1\, b_2 + b_1\, b_3 + b_2\, b_3 + b_1\, b_4 + b_2\, b_4 + b_3\, b_4 \,, \\ a_1\, a_2\, a_3 + a_1\, a_2\, a_4 + a_2\, a_3\, a_4 + a_1\, a_3\, a_4 \le b_1\, b_2\, b_3 + b_1\, b_2\, b_4 + b_2\, b_3\, b_4 + b_1\, b_3\, b_4 \,, \\ a_1\, a_2\, a_3\, a_4 = b_1\, b_2\, b_3\, b_4 \,. \end{gathered}$$

Then

$$\log^2 a_1 + \log^2 a_2 + \log^2 a_3 + \log^2 a_4 \le \log^2 b_1 + \log^2 b_2 + \log^2 b_3 + \log^2 b_4 \,.$$

Proof. Use Corollary 3.7 and observe that S may be an arbitrary subset of $\{1, 2, 3\}$. ■

Corollary 3.9
Let $n \ge 2$ *be an integer and let* T *be an arbitrary subset of* $\{1, 2, \ldots, n-1\}$. *Assume that the conjecture holds for* n *and for any nonempty subset* S *of* T. *Let moreover* $f \in C^n(0, \infty)$. *Then the inequality*

$$f(a_1) + f(a_2) + \ldots + f(a_n) \le f(b_1) + f(b_2) + \ldots + f(b_n)$$

holds for all $a, b \in \Delta_n$ *satisfying*

$$E_k(a) \le E_k(b) \text{ for } k \in T \quad \text{and} \quad E_k(a) = E_k(b) \text{ for } k = 0 \text{ or } k \notin T \tag{3.8}$$

if and only if

$$(-1)^{n+k} (x^k f'(x))^{(n-1)} \le 0 \quad \text{for all } x > 0 \text{ and all } k \in T \,. \tag{3.9}$$

Proof. Assume first (3.9) holds and let $a, b \in \Delta_n$ satisfy (3.8). Consider any $c \in \Delta_n$ with $a \le c < b$. Then the pair c, b satisfies condition (2.4) for some nonempty subset S of T. Therefore by Proposition 3.4, the set $\mathcal{C}(c, b)$ is nonempty and hence by Proposition 3.2, $a \preceq b$. Now Theorem 2.3 implies that inequality (2.6) holds.

Conversely, if (2.6) holds for all $a, b \in \Delta_n$ satisfying (3.8), then (2.6) also holds for all $a, b \in \Delta_n$ satisfying condition (2.4) with $S = T$. Thus Theorem 2.4 implies (3.9). This completes the proof. ■

4 Outlook

Our result generalizes and extents the previously known results on the sum of squared logarithms inequality. Indeed, compared to the proof in [1] our development here views the problem from a different angle in that it is not the logarithm function that defines the problem, but a certain monotonicity property in the geometry of polynomials, explicitly stated in Conjecture 3.3.

If one tries to adopt the above proof of Conjecture 3.3 for $n \le 4$ to the case $n \ge 5$, one has to deal with approximately 2^n cases considered separately. Therefore it is clear, that the extension to natural numbers n beyond $n = 6$, say, is out of reach with such a method. Instead, a general argument should be found to prove or disprove Conjecture 3.3 for general n. Furthermore, it might be worthwhile to develop a better understanding of the differential inequality condition $(-1)^{n+k}(x^k f'(x))^{(n-1)} \le 0$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed fully to all parts of this paper.

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References

[1] M. Bîrsan, P. Neff, and J. Lankeit. Sum of squared logarithms - An inequality relating positive definite matrices and their matrix logarithm. *J. Inequalities Appl.2013*, DOI: 10.1186/1029-242X-2013-168:2013:168, 2013.

[2] J. Lankeit, P. Neff, and Y. Nakatsukasa. The minimization of matrix logarithms - on a fundamental property of the unitary polar factor. *J. Lin. Alg. Appl.*, 449:28–42, 2014.

[3] P. Neff, B. Eidel, F. Osterbrink, and R. Martin. A Riemannian approach to strain measures in nonlinear elasticity. *C. R. Acad. Sci. Paris (Mecanique)*, 342(4):254–257, 2014.

[4] P. Neff, I.D. Ghiba, J. Lankeit, R. Martin, and D. Steigmann. The exponentiated Hencky-logarithmic strain energy. Part II: Coercivity, planar polyconvexity and existence of minimizers. *submitted to Z. Angew. Math. Phys., ArXiv: 1408.4430v1*, 2014.

[5] P. Neff, J. Lankeit, and I.D. Ghiba. The exponentiated Hencky-logarithmic strain energy. Part I: Constitutive issues and rank-one convexity. *submitted to J. Elast., ArXiv: 1403.3843*, 2014.

[6] P. Neff, J. Lankeit, and A. Madeo. On Grioli's minimum property and its relation to Cauchy's polar decomposition. *Int. J. Engng. Sci.*, 80:209–217, 2014.

[7] P. Neff, Y. Nakatsukasa, and A. Fischle. A logarithmic minimization property of the unitary polar factor in the spectral norm and the Frobenius matrix norm. *SIAM J. Matrix Analyis, ArXiv: 1302.3235v4*, 35:1132–1154, 2014.