On the role of micro-inertia in enriched continuum mechanics

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Abstract

In this paper the role of gradient micro-inertia terms $\overline{\eta}\|\nabla u_{,t}\|^2$ and free micro-inertia terms $\eta\|P_{,t}\|^2$ is investigated to unveil their respective effect on the dynamical behavior of band-gap metamaterials. We show that the term $\overline{\eta}\|\nabla u_{,t}\|^2$ alone is only able to disclose relatively simplified dispersive behaviors. On the other hand, the term $\eta\|P_{,t}\|^2$ is in charge of the description of the full complex behavior of band-gap metamaterials. A suitable mixing of the two micro-inertia terms allows to describe a new feature of the relaxed-micromorphic model, i.e. the description of a second band-gap occurring for higher frequencies. We also show that a split of the gradient micro-inertia $\overline{\eta}\|\nabla u_{,t}\|^2$, in the sense of Cartan-Lie decomposition of matrices, allows to flatten separately longitudinal and transverse optic branches thus giving the possibility of a second band-gap. Finally, we investigate the effect of the gradient inertia $\overline{\eta}\|\nabla u_{,t}\|^2$ on more classical enriched models as the Mindlin-Eringen and the internal variable ones. We find that the addition of such gradient micro-inertia allows for the onset of one band-gap in the Mindlin-Eringen model and of three band-gaps in the internal variable model. In this last case, however, non-local effects cannot be accounted for which is a too drastic simplification for most metamaterials. We conclude that, even when adding gradient micro-inertia terms, the relaxed micromorphic model remains the most performing one, among the considered enriched model, for the description of non-local band-gap metamaterials.

Keywords: gradient micro-inertia, free micro-inertia, complete band-gaps, non-local effects, relaxed micromorphic model, generalized continuum models, multi-scale modeling

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Contents

1 Introduction

The question of effectively studying the dynamical behavior of microscopically heterogeneous materials in the simplified framework of continuum mechanics is a major challenge for engineering sciences.

Indeed, it is rather clear at the present state of knowledge that classical Cauchy continuum models are too simplified to describe the behavior of a huge class of materials in the dynamical regime. As a matter of fact, almost all real materials show dispersive behaviors with respect to wave propagation, especially when considering waves with small wavelengths (higher frequencies). More precisely, this means that the speed of propagation of waves is not a constant, as it happens for Caychy continua, but it depends on the wavelength of the traveling wave. Such phenomenon is rather comprehensible if one thinks to the fact that the mechanical properties of materials vary when going down to lower scales. It is then sensible that the speed of propagation of mechanical waves varies when considering waves with wavelength which are small enough to be comparable to the characteristic size of the underlying heterogeneities.

If Cauchy continuum theories are not rich enough to catch these dispersive behaviors, generalized continuum theories have disclosed the possibility of describing wave dispersion while still remaining in the framework of continuum mechanics. Although various generalized continuum models have been introduced to describe dispersion (see, among others, the pioneering works [6, 14]), it is yet not completely clear whether such dispersive properties can be attributed to the constitutive assumptions which are made on the strain energy density or to the choice of the micro-inertia terms which can be introduced.

The aforementioned considerations about the dispersive behavior of materials can be reformulated with renewed awareness when talking about metamaterials.

Metamaterials are man-made artifacts which are conceived by assembling small structural elements in periodic or quasi-periodic patterns in such a way that the resulting material shows new astonishing mechanical properties. The characteristic size of microstructures in such metamaterials is much higher than the characteristic size of heterogeneities in more classical materials. In fact, metamaterials' microstructures usually have characteristic sizes ranging from microns to centimeters, so that it is not necessary to go down to the molecular or the atomic scale to be aware of their discreteness. It is thus not astonishing that metamaterials start showing dispersive behaviors for wavelengths which are much higher than those needed to unveil dispersion in classical materials.

More than this, some metamaterials can exhibit dynamical behaviors which are by far more complex than the simple dispersion. For example, some metamaterials are able to inhibit wave propagation within certain frequency ranges due to the presence of an underlying microstucture which is able to resonate locally when excited at those frequencies or even to remain completely unperturbed. The energy of the incident wave remains trapped at the level of the microstructure and the macroscopic propagation results to be inhibited [1, 13, 21, 22].

In order to catch the complex behavior exhibited by such metamaterials while remaining in the framework of continuum mechanics, generalized continuum models with enriched kinematics are needed. This means that extra degrees of freedom must be introduced in the spirit of micromorphic theories [6, 14] which allow to account for micro-motions at the level of the microstructure. More particularly, the kinematical unknowns of such micromorphic models are usually the macro-displacements u and the micro-distortion tensor P. Well adapted constitutive choices must then be introduced for the strain energy density in order to well describe the behavior of the considered metamaterials in the static regime.

As a last point, the inertia of the considered continuum must be introduced to model its mechanical behavior in the dynamic regime. It is exactly this point that will be the focus of the present paper: how to choose well-suited micro-inertia terms when dealing with enriched continuum models? How each of these terms affects the dynamic behavior of real band-gap metamaterials? Some hints on the role of micro-inertia to model dispersive behaviors are given in [2] but many fundamental questions still remain open.

We will show in this paper that:

- Gradient micro-inertia terms $\overline{\eta}\,\|\,\nabla u_{\,,t}\|^2$ only allow to describe dispersion either in classical or enriched continuum models [2] ,

- Micro-inertia terms involving time derivatives of the extra kinematical degrees of freedom $\eta\,\|P_{,t}\|^2$ allow to describe and control optic branches in the dispersion relations of classical and relaxed micromorphic continuum models [6–12, 14–16],

- The relaxed micromorphic model with micro-inertia of the type $\eta\,\|P_{,t}\|^2$ is able to describe the onset of the first band-gaps in mechanical metamaterials [8–12],

- The relaxed micromorphic model with both micro-inertia terms $\eta\,\|P_{,t}\|^2$ and $\overline{\eta}\,\|\,\nabla u_{\,,t}\|^2$ allows to account for the first and also for the second band-gap which occurs for higher frequencies,

- Classical Mindlin-Eringen models with full micro-inertia $\eta\,\|P_{,t}\|^2$ and $\overline{\eta}\,\|\,\nabla u_{\,,t}\|^2$ allow for the description of only the first band-gap.

- Internal variable models with full micro-inertia $\eta\,\|P_{,t}\|^2$ and $\overline{\eta}\,\|\,\nabla u_{\,,t}\|^2$ allow for the description of three band-gaps, even if some peculiar phenomena related to non-locality cannot be accounted for and the behavior thus results to be too simplified to model realistic metamaterials.

Finally, we show that a weighted gradient micro-inertia of the type $\frac{1}{2}\overline{\eta}_1\,\|\,\mathrm{dev\,sym}\,\nabla u_{\,,t}\|^2+\frac{1}{2}\overline{\eta}_2\,\|\,\mathrm{skew}\,\nabla u_{\,,t}\|^2+\frac{1}{6}\overline{\eta}_3\,\mathrm{tr}\,(\,\nabla u_{\,,t})^2$ allows to flatten some optic curves independently for longitudinal and transverse waves. More precisely, if the parameter $\overline{\eta}_3$ allows to flatten one optic curve for longitudinal waves, the parameter $\overline{\eta}_2$ has an analogous effect for transverse waves. Such improved control on the dispersion curves will allow for a more effective fitting procedure on real band-gap metamaterials, since the description of the second band-gap occurring at higher frequencies becomes accessible.

2 The relaxed micromorphic model

Our novel relaxed micromorphic model endows Mindlin-Eringen's representation with the second order **dislocation density tensor** $\alpha = -\,\mathrm{Curl}P$ instead of the full gradient ∇P.[6] In the isotropic case the elastic energy reads

[6]The dislocation tensor is defined as $\alpha_{ij} = -\,(\,\mathrm{Curl}P)_{ij} = -P_{ih,k}\epsilon_{jkh}$, where ϵ is the Levi-Civita tensor and Einstein notation of sum over repeated indices is used.

$$W = \underbrace{\mu_e \, \| \operatorname{sym} (\nabla u - P)\|^2 + \frac{\lambda_e}{2} (\operatorname{tr} (\nabla u - P))^2}_{\text{isotropic elastic} - \text{energy}} + \underbrace{\mu_c \, \| \operatorname{skew} (\nabla u - P)\|^2}_{\text{rotational elastic coupling}} \tag{1}$$

$$+ \underbrace{\mu_{\text{micro}} \, \| \operatorname{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2} (\operatorname{tr} P)^2}_{\text{micro} - \text{self} - \text{energy}} + \underbrace{\frac{\mu_e L_c^2}{2} \, \| \operatorname{Curl} P\|^2}_{\text{isotropic curvature}} \ ,$$

where the parameters and the elastic stress are analogous to the standard Mindlin-Eringen micromorphic model. The model is well-posed in the statical and dynamical case including when $\mu_c = 0$, see [7,15].

In our relaxed model the complexity of the general micromorphic model has been decisively reduced featuring basically only symmetric gradient micro-like variables and the Curl of the micro-distortion P. However, the relaxed model is still general enough to include the full micro-stretch as well as the full Cosserat micro-polar model, see [16]. Furthermore, well-posedness results for the statical and dynamical cases have been provided in [16] making decisive use of recently established new coercive inequalities, generalizing Korn's inequality to incompatible tensor fields [4,5,18–20].

The relaxed micromorphic model counts 6 constitutive parameters in the isotropic case (μ_e, λ_e, μ_{micro}, λ_{micro}, μ_c, L_c). The characteristic length L_c is intrinsically related to non-local effects due to the fact that it weights a suitable combination of first order space derivatives in the strain energy density (1). For a general presentation of the features of the relaxed micromorphic model in the anisotropic setting, we refer to [3].

As for the kinetic energy, we consider in this paper that it takes the following form:

$$J = \underbrace{\frac{1}{2}\rho \, \|u_{,t}\|^2}_{\text{Cauchy inertia}} + \underbrace{\frac{1}{2}\eta \, \|P_{,t}\|^2}_{\text{free micro-inertia}} + \underbrace{\frac{1}{2}\overline{\eta}_1 \, \| \operatorname{dev} \operatorname{sym} \nabla u_{,t}\|^2 + \frac{1}{2}\overline{\eta}_2 \, \| \operatorname{skew} \nabla u_{,t}\|^2 + \frac{1}{6}\overline{\eta}_3 \operatorname{tr} (\nabla u_{,t})^2}_{\text{new gradient micro-inertia}}, \tag{2}$$

where ρ is the value of the average macroscopic mass density of the considered metamaterial, η is the free micro-inertia density and the $\overline{\eta}_i, i = \{1,2,3\}$ are the gradient micro-inertia densities associated to the different terms of the Cartan-Lie decomposition of ∇u.

The associated equations of motion in strong form, obtained by a classical least action principle take the form (see [10–12,15])

$$\rho \, u_{,tt} + \underbrace{\operatorname{Div}[\mathcal{I}]}_{\text{new augmented term}} = \operatorname{Div} [\widetilde{\sigma}] , \qquad \eta \, P_{,tt} = \widetilde{\sigma} - s - \operatorname{Curl} m, \tag{3}$$

where

$$\begin{aligned}
\mathcal{I} &= \overline{\eta}_1 \operatorname{dev} \operatorname{sym} \nabla u_{,tt} + \overline{\eta}_2 \operatorname{skew} \nabla u_{,tt} + \frac{1}{3}\overline{\eta}_3 \operatorname{tr} (\nabla u_{,tt}) , \\
\widetilde{\sigma} &= 2 \, \mu_e \operatorname{sym} (\nabla u - P) + \lambda_e \operatorname{tr} (\nabla u - P) \, \mathbb{1} + 2 \, \mu_c \operatorname{skew} (\nabla u - P) , \\
s &= 2 \, \mu_{\text{micro}} \operatorname{sym} P + \lambda_{\text{micro}} \operatorname{tr} (P) \, \mathbb{1}, \\
m &= \mu_e L_c^2 \operatorname{Curl} P.
\end{aligned} \tag{4}$$

The fact of adding a gradient micro-inertia in the kinetic energy (2) modifies the strong-form PDEs of the relaxed micromorphic model with the addition of the new term $\mathcal{I}$. Of course, boundary conditions would also be modified with respect to the ones presented in [8,12]. The study of the new boundary conditions induced by gradient micro-inertia will be the object of a subsequent paper where the effect of such extra terms on the conservation of energy will also be analyzed.

3 Plane wave propagation

Sufficiently far from a source, dynamic wave solutions may be treated as planar waves. Therefore, we now want to study harmonic solutions traveling in an infinite domain for the differential system (3). We suppose that the space dependence of all introduced kinematical fields are limited to the scalar component X which is also the direction of propagation of the wave. To do so, following [8–12, 17] we define:

$$
\begin{aligned}
P^S &:= \frac{1}{3}\,\mathrm{tr}\,(P)\,, & P_{[ij]} &:= (\,\mathrm{skew} P)_{ij} = \frac{1}{2}\,(P_{ij} - P_{ji})\,, \\
P^D &:= P_{11} - P^S, & P_{(ij)} &:= (\,\mathrm{sym}\, P)_{ij} = \frac{1}{2}\,(P_{ij} + P_{ji})\,, \\
P^V &:= P_{22} - P_{33}.
\end{aligned}
\tag{5}
$$

With this decomposition, equations (3) can be rewritten as (see [10, 11])

- a set of three equations only involving longitudinal quantities:

$$
\begin{aligned}
\rho\,\ddot{u}_1 - \underbrace{\frac{2\,\overline{\eta}_1 + \overline{\eta}_3}{3}\,\ddot{u}_{1,11}}_{\text{new augmented terms}} &= (2\,\mu_e + \lambda_e)\,u_{1,11} - 2\mu_e\,P^D_{,1} - (2\mu_e + 3\lambda_e)\,P^S_{,1}\,, \\
\eta\,\ddot{P}^D &= \frac{4}{3}\,\mu_e\,u_{1,1} + \frac{1}{3}\,\mu_e L_c^2\,P^D_{,11} - \frac{2}{3}\,\mu_e L_c^2 P^S_{,11} - 2\,(\mu_e + \mu_{\mathrm{micro}})\;P^D\,, \\
\eta\,\ddot{P}^S &= \frac{2\,\mu_e + 3\,\lambda_e}{3}\,u_{1,1} - \frac{1}{3}\,\mu_e L_c^2 P^D_{,11} + \frac{2}{3}\,\mu_e L_c^2 P^S_{,11} \\
&\quad - (2\,\mu_e + 3\,\lambda_e + 2\,\mu_{\mathrm{micro}} + 3\,\lambda_{\mathrm{micro}})\;P^S\,,
\end{aligned}
\tag{6}
$$

- two sets of three equations only involving transverse quantities in the ξ-th direction, with $\xi = 2, 3$:

$$
\begin{aligned}
\rho\,\ddot{u}_\xi - \underbrace{\frac{\overline{\eta}_1 + \overline{\eta}_2}{2}\,\ddot{u}_{\xi,11}}_{\text{new augmented terms}} &= (\mu_e + \mu_c)\,u_{\xi,11} - 2\,\mu_e\,P_{(1\xi),1} + 2\,\mu_c\,P_{[1\xi],1}, \\
\eta\,\ddot{P}_{(1\xi)} &= \mu_e\,u_{\xi,1} + \frac{1}{2}\,\mu_e L_c^2\,P_{(1\xi),11} + \frac{1}{2}\,\mu_e L_c^2\,P_{[1\xi],11} \\
&\quad - 2\,(\mu_e + \mu_{\mathrm{micro}})\;P_{(1\xi)}, \\
\eta\,\ddot{P}_{[1\xi]} &= -\mu_c\,u_{\xi,1} + \frac{1}{2}\,\mu_e L_c^2\,P_{(1\xi),11} + \frac{1}{2}\,\mu_e L_c^2 P_{[1\xi],11} - 2\,\mu_c\,P_{[1\xi]},
\end{aligned}
\tag{7}
$$

- One equation only involving the variable $P_{(23)}$:

$$
\eta\,\ddot{P}_{(23)} = -2\,(\mu_e + \mu_{\mathrm{micro}})\,P_{(23)} + \mu_e L_c^2 P_{(23),11}, \tag{8}
$$

- One equation only involving the variable $P_{[23]}$:

$$
\eta\,\ddot{P}_{[23]} = -2\,\mu_c\,P_{[23]} + \mu_e L_c^2 P_{[23],11}, \tag{9}
$$

- One equation only involving the variable P^V:

$$
\eta\,\ddot{P}^V = -2\,(\mu_e + \mu_{\mathrm{micro}})\,P^V + \mu_e L_c^2 P^V_{,11}. \tag{10}
$$

Once this symplified system of PEDEs is obtained, we look for a wave form solution of the type:

$$
\underbrace{\mathbf{v}_1(X,t) = \boldsymbol{\beta}\,e^{i(kX - \omega t)}}_{\text{longitudinal}}, \qquad \underbrace{\mathbf{v}_\tau(X,t) = \boldsymbol{\gamma}^{\,\tau} e^{i(kX - \omega t)}}_{\text{transversal}}, \quad \tau = 2, 3, \qquad \underbrace{\mathbf{v}_4(X,t) = \boldsymbol{\gamma}^{\,4} e^{i(kX - \omega t)}}_{\text{uncoupled}}, \tag{11}
$$

where we set for compactness

$$
\mathbf{v}_1 = \left(u_1, P^D, P^S\right) \qquad \mathbf{v}_\tau = \left(u_\tau, P_{(1\tau)}, P_{[1\tau]}\right), \quad \tau = 2, 3, \qquad \mathbf{v}_4 = \left(P_{(23)}, P_{[23]}, P^V\right). \tag{12}
$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T \in \mathbb{C}^3$, $\boldsymbol{\gamma}^\tau = (\gamma_1^\tau, \gamma_2^\tau, \gamma_3^\tau)^T \in \mathbb{C}^3$ and $\boldsymbol{\gamma}^4 = (\gamma_1^4, \gamma_2^4, \gamma_3^4)^T \in \mathbb{C}^3$ are the unknown amplitudes of the considered waves[7], k is the wavenumber and ω is the wave-frequency.

Replacing the wave form solution (11) in Eqs. (6), (7), (8), (9) and (10), it is possible to express the system as:

$$\mathbf{A}_1 \cdot \boldsymbol{\beta} = 0, \qquad \mathbf{A}_\tau \cdot \boldsymbol{\gamma}^\tau = 0, \quad \tau = 2,3, \qquad \mathbf{A}_4 \cdot \boldsymbol{\gamma}^4 = 0, \tag{13}$$

where

$$\mathbf{A}_1(\omega,k) = \begin{pmatrix} -\omega^2\left(1 + k^2\,\frac{2\,\overline{\eta}_1+\overline{\eta}_3}{3\,\rho}\right) + c_p^2\,k^2 & i\,k\,2\,\mu_e/\rho & i\,k\,\left(2\,\mu_e + 3\,\lambda_e\right)/\rho \\ -i\,k\,\frac{4}{3}\,\mu_e/\eta & -\omega^2 + \frac{1}{3}k^2 c_m^2 + \omega_s^2 & -\frac{2}{3}\,k^2 c_m^2 \\ -\frac{1}{3}\,i\,k\,\left(2\,\mu_e + 3\,\lambda_e\right)/\eta & -\frac{1}{3}\,k^2\,c_m^2 & -\omega^2 + \frac{2}{3}\,k^2\,c_m^2 + \omega_p^2 \end{pmatrix},$$

$$\begin{gathered} \mathbf{A}_2(\omega,k) = \mathbf{A}_3(\omega,k) = \begin{pmatrix} -\omega^2\left(1 + k^2\,\frac{\overline{\eta}_1+\overline{\eta}_2}{2\,\rho}\right) + k^2 c_s^2 & i\,k\,2\,\mu_e/\rho & -i\,\frac{k}{\rho}\eta\,\omega_r^2, \\ -\,i\,k\,\mu_e/\eta, & -\omega^2 + \frac{c_m^2}{2}k^2 + \omega_s^2 & \frac{c_m^2}{2}k^2 \\ \frac{i}{2}\,\omega_r^2\,k & \frac{c_m^2}{2}k^2 & -\omega^2 + \frac{c_m^2}{2}k^2 + \omega_r^2 \end{pmatrix}, \\ \mathbf{A}_4(\omega,k) = \begin{pmatrix} -\omega^2 + c_m^2\,k^2 + \omega_s^2 & 0 & 0 \\ 0 & -\omega^2 + c_m^2\,k^2 + \omega_r^2 & 0 \\ 0 & 0 & -\omega^2 + c_m^2\,k^2 + \omega_s^2 \end{pmatrix}. \end{gathered} \tag{14}$$

Here, we have defined:

$$\boxed{\begin{array}{lll} c_m = \sqrt{\dfrac{\mu_e L_c^2}{\eta}}, & c_s = \sqrt{\dfrac{\mu_e + \mu_c}{\rho}}, & c_p = \sqrt{\dfrac{2\,\mu_e + \lambda_e}{\rho}}, \\ \omega_s = \sqrt{\dfrac{2\,(\mu_e + \mu_{\rm micro})}{\eta}}, & \omega_p = \sqrt{\dfrac{(2\,\mu_e + 3\,\lambda_e) + (2\,\mu_{\rm micro} + 3\,\lambda_{\rm micro})}{\eta}}, & \omega_r = \sqrt{\dfrac{2\,\mu_c}{\eta}}, \end{array}}$$

In order to have non-trivial solutions of the algebraic systems (13), one must impose that

$$\underbrace{\det \mathbf{A}_1(\omega,k) = 0,}_{\text{longitudinal}} \qquad \underbrace{\det \mathbf{A}_2(\omega,k) = \det \mathbf{A}_3(\omega,k) = 0,}_{\text{transverse}} \qquad \underbrace{\det \mathbf{A}_4(\omega,k) = 0,}_{\text{uncoupled}} \tag{15}$$

The solutions $\omega = \omega(k)$ of these algebraic equations are called the dispersion curves of the relaxed micromorphic model for longitudinal, transverse and uncoupled waves, respectively.

In what follows we will present the results obtained for the numerical values of the elastic coefficients chosen as in Table 1 if not differently specified.

[7] Here, we understand that having found the (in general, complex) solutions of (11) only the real or imaginary parts separately constitute actual wave solutions which can be observed in reality.

Parameter	Value	Unit
μ_e	200	MPa
$\lambda_e = 2\mu_e$	400	MPa
$\mu_c = 5\mu_e$	1000	MPa
μ_{micro}	100	MPa
λ_{micro}	100	MPa
L_c	1	mm
ρ	2000	kg/m^3

Parameter	Value	Unit
λ_{macro}	82.5	MPa
μ_{macro}	66.7	MPa
E_{macro}	170	MPa
ν_{macro}	0.28	–

Table 1: Values of the parameters used in the numerical simulations (left) and corresponding values of the Lamé parameters and of the Young modulus and Poisson ratio (right), for the formulas needed to calculate the homogenized macroscopic parameters starting from the microscopic ones, see [3].

In the following sections we will explicitly discuss which is the effect of each micro-inertia parameter on the dispersion curves of the relaxed micromorphic model. More particularly, we will focus on the cases:

- vanishing free micro-inertia $\eta = 0$ and non-vanishing gradient micro-inertia,
- both non-vanishing gradient micro-inertia and free micro-inertia.

The remaining case (vanishing gradient micro-inertia $\overline{\eta} = 0$ and non-vanishing free micro-inertia $\eta \neq 0$) is the classical case treated for the relaxed micromorphic model in [8–12, 17]. To the sake of completeness, we present here again the dispersion curves for this case when using the values of the parameters given in Tab. 1.

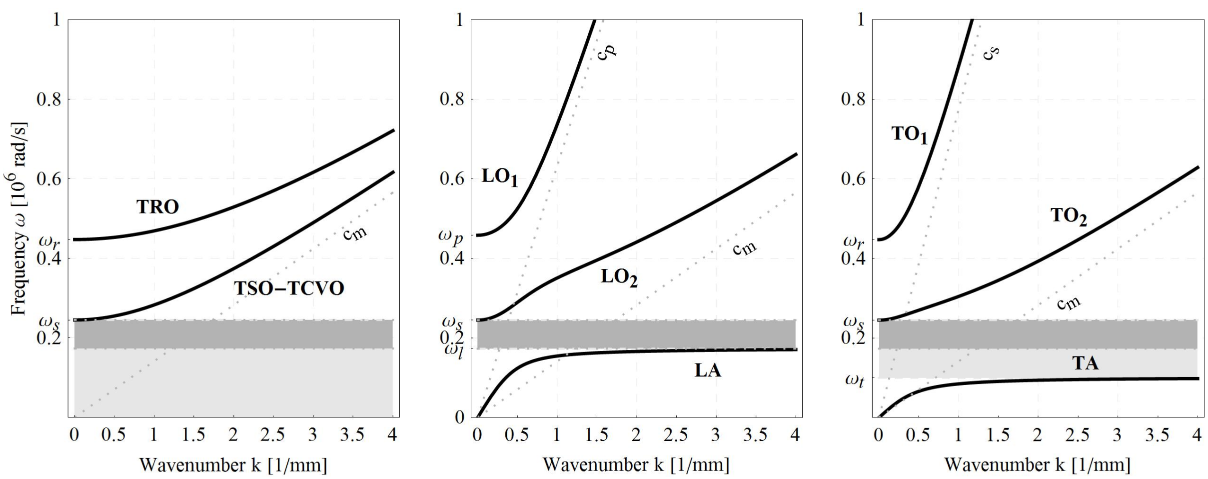

Figure 1: Dispersion relations $\omega = \omega(k)$ for the uncoupled (left), longitudinal (center) and transverse (right) waves of the **relaxed micromorphic model** with free micro-inertia $\eta = 10^{-2} kg/m$.

It can be found that, when considering the free micro-inertia alone, the relaxed micromorphic model is able to predict the first band gap which usually occurs at relatively low frequencies. Moreover, the relaxed micromorphic model is, to the current state of the art, the only continuum model which is able to describe simultaneously band-gaps and non-local behavior [8].

In the next sections we will present the new results concerning the effect of the gradient micro-inertia terms on the dispersion curves of the relaxed micromorphic model, as well as the effect of such gradient micro-inertia on more classical enriched models (Mindlin, internal variable).

4 Case of vanishing free micro-inertia η and non-vanishing gradient micro-inertia $\overline{\eta}$

In this section we discuss the effect, on the dispersion curves of enriched continuum models, of the gradient micro-inertia term alone. We will show that the fact of complementing the macro-inertia $\rho\|u_{,t}\|^2$ only with the gradient micro-inertia $\overline{\eta}\|\nabla u_{,t}\|^2$ is a fundamental modeling limitation since the complexity of the dynamical behavior of micromorphic models cannot be unveiled. Nevertheless, the gradient micro-inertia allows to describe some dispersion which is not allowed by classical Cauchy models.

4.1 Study of the dispersion curves

In the case in which we consider only the gradient micro-inertia $\overline{\eta} \neq 0$ to be non-vanishing, the matrix associated to the longitudinal dynamical system can be expressed as[8]:

$$\mathbf{A}_1(\omega,k) = \begin{pmatrix} -\omega^2\left(\rho + k^2\,\frac{2\,\overline{\eta}_1+\overline{\eta}_3}{3}\right) + (2\,\mu_e+\lambda_e)\,k^2 & i\,k\,2\mu_e & i\,k\,\left(2\,\mu_e+3\,\lambda_e\right) \\ -i\,k\,\frac{4}{3}\,\mu_e & \frac{1}{3}k^2\mu_e L_c^2 + 2\,(\mu_e+\mu_{\text{micro}}) & -\frac{2}{3}\,k^2\mu_e L_c^2 \\ -\frac{1}{3}\,i\,k\,\left(2\,\mu_e+3\,\lambda_e\right) & -\frac{1}{3}\,k^2\,\mu_e L_c^2 & \frac{2}{3}\,k^2\,\mu_e L_c^2+\omega_p^2 \end{pmatrix}. \tag{16}$$

It is possible to remark that the polynomial $\det \mathbf{A}_1(\omega,k)$ is of the second order in ω. This implies that we have a unique positive solution of the equation $\det \mathbf{A}_1(\omega,k)=0$ when considering positive k [9]. In particular, when plotting such solution in the (ω,k) plane only one acoustic branch can be detected (see Fig. 2)[10].

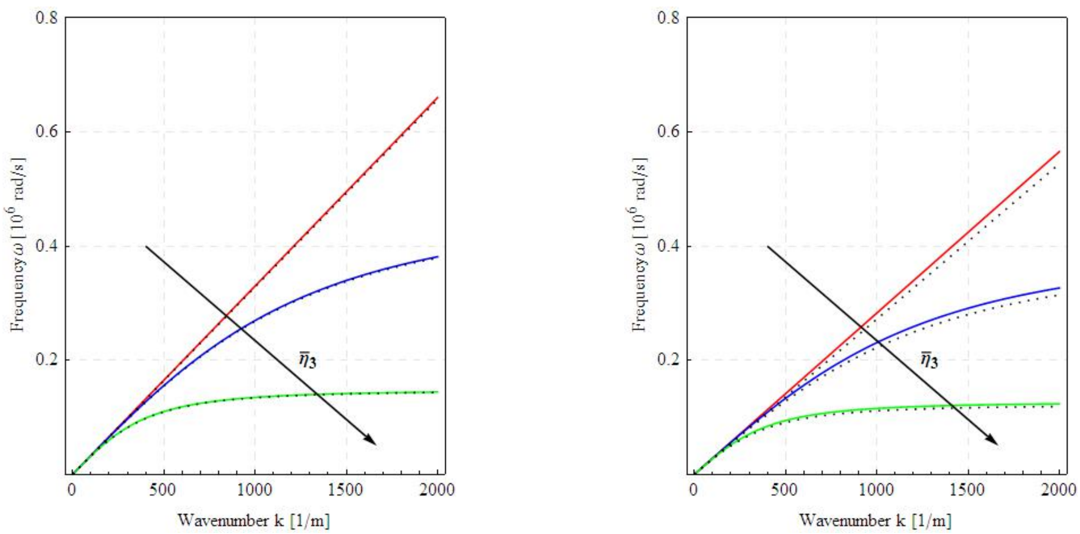

Figure 2: Dispersion relations $\omega=\omega(k)$ for the longitudinal waves of the **relaxed micromorphic model** with gradient micro-inertia $\overline{\eta}_3 = (0, 3\times10^{-3}, 3\times10^{-2})kg/m$ and $\eta=0$. Dotted in black the dispersion relations for a **first gradient model** with Lamé parameters μ_{macro} and λ_{macro} and the same inertiae ρ and $\overline{\eta}_3$ (left). The same picture obtained imposing $\lambda_{\text{micro}}=0$ (right): a very slight variation with respect to the 1^{st} gradient case can be detected.

[8]We can notice from the form of $\mathbf{A}_1(\omega,k)$ that considering an additional micro-inertia $\overline{\eta}$ is equivalent to defining an average macroscopic density depending on the wavelength as $\rho^*(k)=\rho+k^2\,\overline{\eta}$. The same can be found for the transverse waves.

[9]It can be checked that, when considering elastic parameters which guarantee positive definiteness of the elastic energy the solution $\omega=\omega(k)$ of the characteristic polynomials are always real [17].

[10]Here and in the sequel, we will always set $\overline{\eta}_1=0$, since we could not isolate a characteristic effect of such parameters on the dispersion curves.

Comparing the results shown in Fig. 2 with those presented in Fig. 1, it is immediate to notice that the fact of considering the gradient micro-inertia alone significantly constrains the behavior of the considered enriched continuum. Even if the constitutive expression for the strain energy density W is the same both in Fig. 2 and in Fig. 1 (see Eq. (1)), the fact of using a gradient micro-inertia $\overline{\eta}\| \nabla u_{,t}\|^2$ instead of a free micro-inertia $\eta\|P_{,t}\|^2$ drastically simplifies the patterns which are found for the dispersion curves. With reference to Fig. 2, we can remark that a unique acoustic wave is found and that the presence of a non-vanishing micro-inertia parameter $\overline{\eta}_3$ induces a dispersive behavior. When the gradient micro-inertia parameters are all vanishing ($\overline{\eta}_1 = \overline{\eta}_2 = \overline{\eta}_3 = 0$), this means that only a macro-inertia $\rho\|u_{,t}\|^2$ is present and this correspond to an almost constant speed of the traveling waves, which it is what happens for the classical Cauchy case. It can be shown that, considering an adapted choice of the constitutive parameters for the relaxed micromorphic model with macro-inertia $\rho\|u_{,t}\|^2$ alone, the dispersion curve obtained is exactly the straight one obtained with classical Cauchy model.

With a similar reasoning as the one made for longitudinal waves, considering the case $\overline{\eta} \neq 0$ for transverse waves, the matrix associated to the transverse dynamical system can be expressed as

$$\mathbf{A}_2(\omega,k) = \begin{pmatrix} -\omega^2\left(\rho + k^2\,\frac{\overline{\eta}_1+\overline{\eta}_2}{2}\right) + k^2(\mu_e+\mu_c) & i\,k\,2\mu_e & -i\,k\,2\mu_c \\ -i\,k\,2\mu_e & \mu_e L_c^2 k^2 + 4(\mu_e+\mu_{\text{micro}}) & \mu_e L_c^2 k^2 \\ i\,k\,2\mu_c & \mu_e L_c^2 k^2 & \mu_e L_c^2\,k^2 + 4\mu_c \end{pmatrix}, \tag{17}$$

It is possible to see that the new inertia terms $\overline{\eta}_2$ plays the same role for the transverse waves that was played by $\overline{\eta}_3$ for the longitudinal waves. The results concerning the solutions $\omega = \omega(k)$ of the characteristic equation $\det \mathbf{A}_2(\omega,k) = 0$ are analogous to the case of longitudinal waves, see Fig. 3.

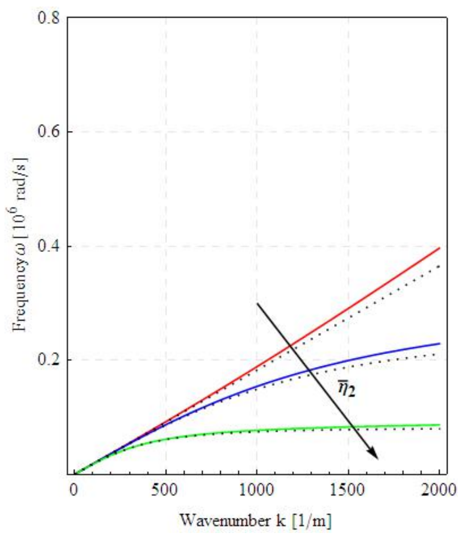

Figure 3: Dispersion relations $\omega = \omega(k)$ for the transverse waves of the **relaxed micromorphic model** with gradient micro-inertia $\overline{\eta}_2 = (0, 2\times10^{-3}, 2\times10^{-2})kg/m$ and $\eta = 0$ and dotted in black the dispersion relations for a **first gradient model** with Lamé parameters μ_{macro} and λ_{macro} and the same inertiae ρ and $\overline{\eta}_t$.

If the particular case with non-null gradient micro-inertia $\overline{\eta} \neq 0$ and null free micro-inertia $\eta = 0$ is considered, the matrix associate to the uncoupled waves reduces to:

$$\mathbf{A}_4(\omega,k) \,=\, \begin{pmatrix} \mu_e L_c^2\, k^2 + 2\,(\mu_e+\mu_{\text{micro}}) & 0 & 0 \\ 0 & \mu_e L_c^2\, k^2 + 2\,\mu_c & 0 \\ 0 & 0 & \mu_e L_c^2\, k^2 + 2\,(\mu_e+\mu_{\text{micro}}) \end{pmatrix}. \tag{18}$$

from which it is not possible to derive any dispersion curve, due to the absence of inertia terms.

4.2 A first conclusion on the effect of gradient micro-inertia on enriched continuum models.

- When considering a macro-inertia term $\rho\|u_{,t}\|^2$ alone, only one acoustic wave is present and the associated dispersion has an almost constant speed of propagation. Such behavior is strongly dictated by the macro-inertia term since the difference on the associated dispersion curves between a simple Cauchy energy $W(\,\nabla u\,)$ and an enriched model $W = W(\,\nabla u\,, P,\, \operatorname{Curl} P)$ is small and vanishing considering an adapted choice of the constitutive parameters.

- When considering a complementation of the macro-inertia $\rho\|u_{,t}\|^2$ with a gradient micro-inertia $\overline{\eta}\|\,\nabla u_{\,,t}\|^2$ the speed of propagation of waves is not constant anymore, but it depends on the wavelength of the traveling waves. Nevertheless, only an acoustic branch can be described, independently of the more or less complicated (standard or enriched) kinematics.

- Complementing the macro-inertia $\rho\|u_{,t}\|^2$ with a free micro-inertia $\eta\|P_{,t}\|^2$ allows to disclose the full rich constitutive behavior provided by the fact of considering an enriched model, as studied in [8–12,17] and reproduced in Fig. 1. Two optic waves are observed, both for longitudinal and transverse waves, in addition to the acoustic ones already discussed in the previous case (see Fig. 1). The properties of such curves depend both on the constitutive parameters appearing in the expression of the energy (Eq. (1)) and on the free inertia parameter η. In this framework of inertia of the type $\rho\|u_{,t}\|^2 + \eta\|P_{,t}\|^2$, the relaxed micromorphic model is the only non-local, enriched continuum model allowing for the presence of band-gaps [9].

5 Case of both non-vanishing free micro-inertia η and gradient micro-inertia $\overline{\eta}$

In this section we will discuss the effect of a full inertia $\rho\|u_{,t}\|^2 + \eta\|P_{,t}\|^2$ on the dispersion curves of the relaxed micromorphic model. We will show that the complementation of the macro inertia with both the gradient and free micro-inertia allows for the description of a new feature of the relaxed micromorphic model, i.e. the onset of a second band-gap occurring at higher frequencies with respect to the first one.

5.1 Dispersion relations

Now, we show in Figure 4 the results obtained for non-null micro-inertia $\eta \neq 0$ with the addition of gradient micro-inertia $\overline{\eta} \neq 0$. Surprisingly, the combined effect of the traditional micro-inertia η with the gradient micro-inertiae can lead to the onset of a second longitudinal and transverse band gap. Moreover, it is possible to notice that the addition of gradient micro-inertiae $\overline{\eta}_1$, $\overline{\eta}_2$ and $\overline{\eta}_3$ has no effect on the cut-off frequencies, which only depend on the free micro-inertia η (and of course on the constitutive parameters).

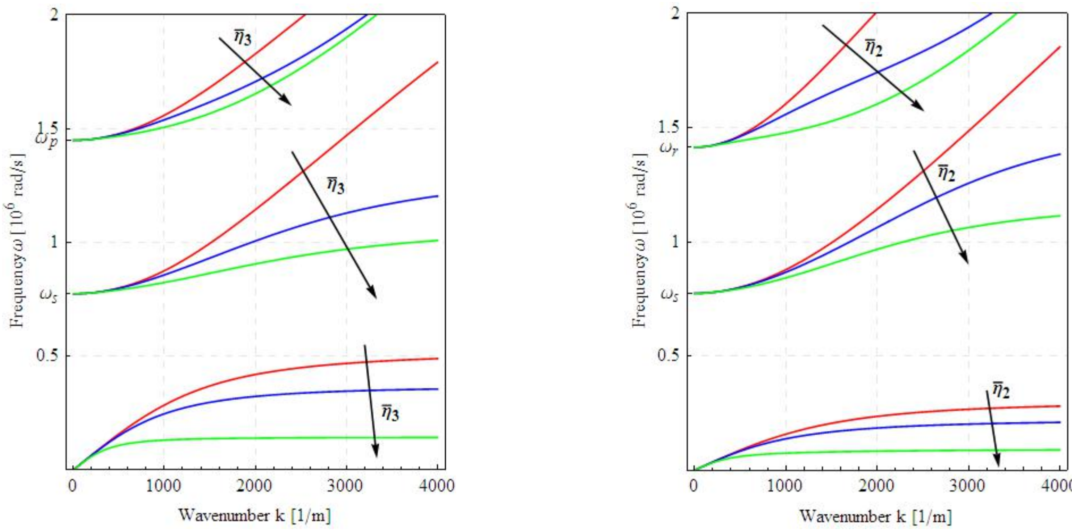

Figure 4: Dispersion relations $\omega = \omega(k)$ of the **relaxed micromorphic model** for the longitudinal waves with free micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\overline{\eta}_3 = (3\times 10^{-4}, 3\times 10^{-3}, 3\times 10^{-2}) kg/m$ (left) and transverse waves with micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\overline{\eta}_2 = (2\times 10^{-4}, 2\times 10^{-3}, 2\times 10^{-2}) kg/m$ (right).

The uncoupled waves in the relaxed micromorphic model with generalized inertia behave as in the relaxed micromorphic model as it is possible to see analyzing the matrix:

$$\mathbf{A}_4(\omega,k) = \begin{pmatrix} -\omega^2 + c_m^2\,k^2 + \omega_s^2 & 0 & 0 \\ 0 & -\omega^2 + c_m^2\,k^2 + \omega_r^2 & 0 \\ 0 & 0 & -\omega^2 + c_m^2\,k^2 + \omega_s^2 \end{pmatrix}. \tag{19}$$

The resulting dispersion curves are the same to the ones obtained with the classical relaxed micromorphic model, see Fig. 1, right.

5.2 Cut-offs and asymptotic behavior

To study the asymptotic behavior of the dispersion curves for the relaxed micromorphic model with full inertia, let us introduce the following quantities:

$$\boxed{\begin{array}{lll} \omega_v = \sqrt{\dfrac{(2\,\mu_e + \lambda_e) + (2\,\mu_{\text{micro}} + \lambda_{\text{micro}})}{\eta}}, & \omega_l = \sqrt{\dfrac{2\,\mu_{\text{micro}} + \lambda_{\text{micro}}}{\eta}}, & \omega_t = \sqrt{\dfrac{\mu_{\text{micro}}}{\eta}}, \\[2ex] \omega_{\overline{l}} = \sqrt{\dfrac{2\,\mu_e + \lambda_e}{\frac{2\,\overline{\eta}_1 + \overline{\eta}_3}{3}}}, & \omega_{\overline{t}} = \sqrt{\dfrac{2\,(\mu_c + \mu_e)}{\frac{\overline{\eta}_1 + \overline{\eta}_2}{2}}}. & \end{array}}$$

As stated in the previous section the cut-off frequencies are not modified by the insertion of a gradient micro-inertia term. Therefore, considering the longitudinal waves, we have one acoustic branch of the dispersion curve and two optic branches with cut-off frequencies:

$$\omega_s = \sqrt{\frac{2\,(\mu_e + \mu_{\text{micro}})}{\eta}}, \qquad \omega_p = \sqrt{\frac{(2\,\mu_e + 3\,\lambda_e) + (2\,\mu_{\text{micro}} + 3\,\lambda_{\text{micro}})}{\eta}}, \tag{20}$$

On the other hand, the asymptotic behavior changes in a radical fashion from the classical relaxed micromorphic model. The horizontal asymptote of the acoustic curve changes and we have the onset of a new horizontal asymptote for one of the optic branches, which values are respectively:

$$\omega_{l,\text{acoustic}} = \sqrt{\frac{\omega_{\overline{l}}^2 + \omega_v^2 - \sqrt{(\omega_{\overline{l}}^2 + \omega_v^2)^2 - 4\,\omega_{\overline{l}}^2\,\omega_l^2}}{2}}, \tag{21}$$
$$\omega_{l,\text{optic}} = \sqrt{\frac{\omega_{\overline{l}}^2 + \omega_v^2 + \sqrt{(\omega_{\overline{l}}^2 + \omega_v^2)^2 - 4\,\omega_{\overline{l}}^2\,\omega_l^2}}{2}}.$$

No difference is found in the other optic branch that has an asymptote with slope c_m as in the classical relaxed micromorphic model.

Analogously, considering the transverse waves, we have one acoustic branch and two optic branches with cut-off frequencies:

$$\omega_s = \sqrt{\frac{2\,(\mu_e + \mu_{\text{micro}})}{\eta}}, \qquad \omega_r = \sqrt{\frac{2\,\mu_c}{\eta}}. \tag{22}$$

Once again, the horizontal asymptote of the acoustic curve changes with respect to the classical relaxed case and we have an extra horizontal asymptote for one of the optic branches, which values are respectively:

$$\omega_{t,\text{acoustic}} = \frac{1}{2}\sqrt{\omega_{\overline{t}}^2 + \omega_s^2 + \omega_r^2 - \sqrt{(\omega_{\overline{t}}^2 + \omega_s^2 + \omega_r^2)^2 - 4\,\omega_{\overline{t}}^2\,\omega_t^2}}, \tag{23}$$
$$\omega_{t,\text{optic}} = \frac{1}{2}\sqrt{\omega_{\overline{t}}^2 + \omega_s^2 + \omega_r^2 + \sqrt{(\omega_{\overline{t}}^2 + \omega_s^2 + \omega_r^2)^2 - 4\,\omega_{\overline{t}}^2\,\omega_t^2}}.$$

No difference is found in the other optic branch that has an asymptote with slope c_m as in the classical relaxed micromorphic model.

Finally, no change whatsoever is present in the uncoupled waves that keep having cut-off frequencies ω_s and ω_r and oblique asymptote of slope c_m.

6 Combined effect of the free and gradient micro-inertiae on more classical enriched models (Mindlin-Eringen and internal variable)

In this section, we discuss the effect on the Mindlin-Eringen and the internal variable model of the addition of the gradient micro-inertia $\overline{\eta}\|\,\nabla u_{,t}\|^2$ to the classical terms $\rho\|u_{,t}\|^2+\eta\|P_{,t}\|^2$. We will show that the previously discussed effect of the parameters $\overline{\eta}_2$ and $\overline{\eta}_3$ is maintained both for the Mindlin-Eringen and for the internal variable case.

Figure 5 refers to the study of the effects of the parameters $\overline{\eta}_2$ and $\overline{\eta}_3$ on the dispersion curves of the classical Mindlin-Eringen micromorphic model. To the sake of completeness, we recall that the (simplified) strain energy density for this model in the isotropic case takes the form:

$$W = \underbrace{\mu_e\,\|\,\text{sym}\,(\,\nabla u\,-P)\|^2 + \frac{\lambda_e}{2}\,(\text{tr}\,(\,\nabla u\,-P))^2}_{\text{isotropic elastic}\,-\,\text{energy}} + \underbrace{\mu_c\,\|\,\text{skew}\,(\,\nabla u\,-P)\|^2}_{\text{rotational elastic coupling}} \tag{24}$$
$$+ \underbrace{\mu_{\text{micro}}\,\|\,\text{sym}\,P\|^2 + \frac{\lambda_{\text{micro}}}{2}\,(\text{tr}P)^2}_{\text{micro}\,-\,\text{self}\,-\,\text{energy}} + \underbrace{\frac{\mu_e L_c^2}{2}\,\|\nabla\,P\|^2}_{\text{isotropic curvature}},$$

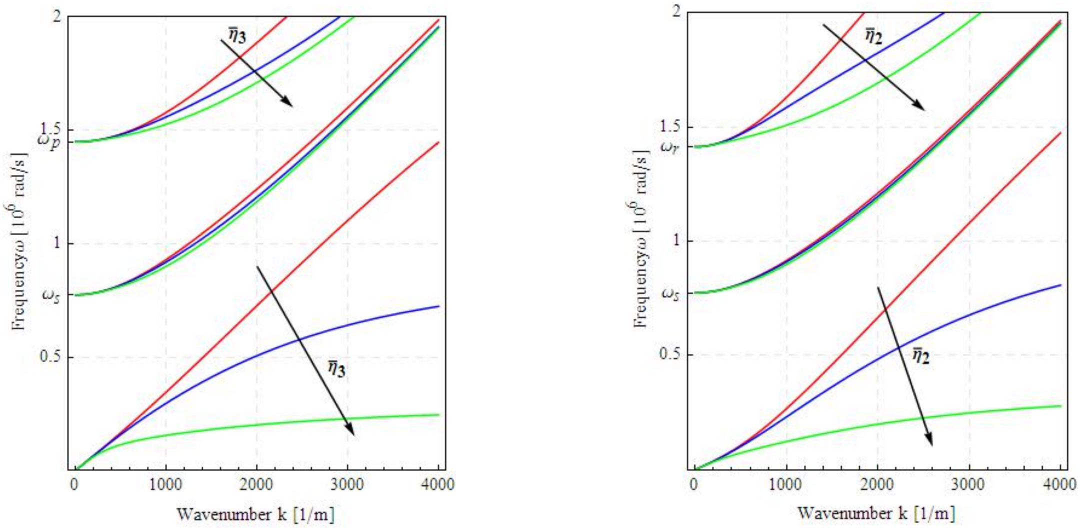

Figure 5: Dispersion relations $\omega = \omega(k)$ of the **standard Mindlin-Eringen model** for the longitudinal waves with free micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\overline{\eta}_3 = (3 \times 10^{-4}, 3 \times 10^{-3}, 3 \times 10^{-2})kg/m$ (left) and transverse waves with micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\overline{\eta}_2 = (2 \times 10^{-4}, 2 \times 10^{-3}, 2 \times 10^{-2})kg/m$ (right).

Recalling the results of [10], we remark that when the gradient micro-inertia is vanishing ($\overline{\eta}_1 = \overline{\eta}_2 = \overline{\eta}_3 = 0$) the Mindlin-Eringen model does not allow the description of band-gaps, due to the presence of a straight acoustic waves. On the other hand, when switching on the parameters $\overline{\eta}_2$ and $\overline{\eta}_3$, the acoustic branches are flattened (they take a horizontal asymptote), so that the first band-gap can be described. The analogous case for the relaxed micromorphic model (Fig. 1) allowed instead for the description of 2 band gaps.

Figure 6 shows the behavior of the addition of the gradient micro-inertia $\overline{\eta}\| \nabla u_{,t}\|^2$ on the internal variable model. We recall (see [16]) that the energy for the internal variable model does not include higher space derivatives of the micro-distortion tensor P and, in the isotropic case, takes the form:

$$W = \underbrace{\mu_e \,\| \operatorname{sym} \left(\nabla u - P\right)\|^2 + \frac{\lambda_e}{2} \left(\operatorname{tr} \left(\nabla u - P\right)\right)^2}_{\text{isotropic elastic} - \text{energy}} + \underbrace{\mu_c \,\| \operatorname{skew} \left(\nabla u - P\right)\|^2}_{\text{rotational elastic coupling}} \tag{25}$$
$$+ \underbrace{\mu_{\text{micro}} \,\| \operatorname{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2} \left(\operatorname{tr} P\right)^2}_{\text{micro} - \text{self} - \text{energy}},$$

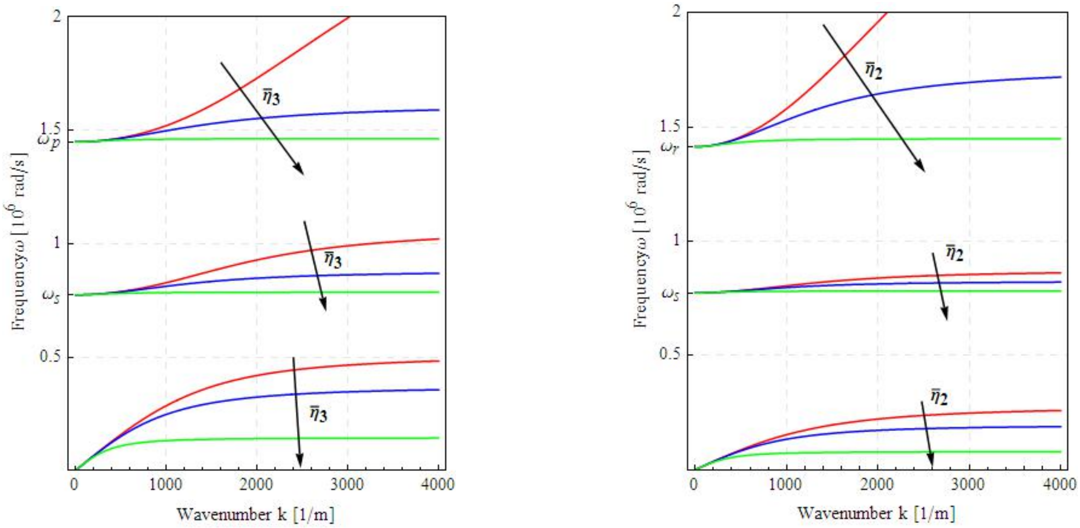

Figure 6: Dispersion relations $\omega = \omega(k)$ of the **internal variable model** for the longitudinal waves with free micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\overline{\eta}_3 = (3\times10^{-4}, 3\times10^{-3}, 3\times10^{-2})kg/m$ (left) and transverse waves with micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\overline{\eta}_2 = (2\times10^{-4}, 2\times10^{-3}, 2\times10^{-2})kg/m$ (right).

By direct observation of Fig. 6, we can notice that suitably choosing the relative position of ω_r and ω_p, the internal variable model allows to account for 3 band gaps.

We thus have an extra band-gap with respect to the analogous case for the relaxed micromorphic model (see Fig. 1), but we are not able to consider non-local effects. The fact of excluding the possibility of describing non-local effects in metamaterials can be sometimes too restrictive. For example, flattening the curve which originates from ω_r and which is associated to rotational modes of the microstructure is unphysical for the great majority of metamaterials.

7 Conclusions

In this paper we discuss the fundamental role of micro-inertia in enriched continuum models of the micromorphic type.

We show that if, on one hand, the free micro-inertia term $\eta\|P_{,t}\|^2$ is strictly necessary to disclose the full rich behavior of micromorphic media in the dynamic regime, on the other hand the gradient micro-inertia $\overline{\eta}\|\nabla u_{,t}\|^2$ has the macroscopic effect of flattening some of the dispersion curves so allowing for the description of extra band-gaps. In particular, we show that:

- In the case of the relaxed micromorphic model one band-gap can be described when introducing the free micro-inertia $\eta\|P_{,t}\|^2$ alone. When introducing a mixed micro-inertia $\eta\|P_{,t}\|^2+\overline{\eta}\|\nabla u_{,t}\|^2$ two band-gaps can be accounted for by the same model.
- In the case of Mindlin-Eringen model no band-gaps are possible with the term $\eta\|P_{,t}\|^2$ alone, while the onset of a single band-gap can be granted by the addition of the extra term $\overline{\eta}\|\nabla u_{,t}\|^2$.
- In the internal variable model two band-gaps are possible with the term $\eta\|P_{,t}\|^2$ alone, even if non-localities cannot be accounted for by such model. When adding the extra term $\overline{\eta}\|\nabla u_{,t}\|^2$ even three band-gaps become possible, but the behavior of the dispersion curves becomes fairly unrealistic for a huge class of real metamaterials.

In conclusion, the results presented in this paper confirm the preceding findings according to wich the relaxed micromorphic model is the most suitable enriched model for the simultaneous description of i) band-gaps and ii) non-localities in mechanical metamaterials.

Future work will be devoted to the application of the results obtained in this paper for the fitting of the proposed model with enriched micro-inertia on real metamaterials exhibiting multiple band-gaps.

8 Acknowledgments

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