Some thoughts on the shear problem

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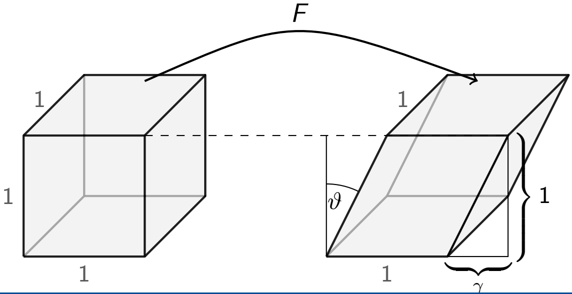
Simple shear deformation [Thiel, Voss, Martin, and Neff 2018b]

A **simple shear deformation** is a mapping $\varphi\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$\nabla\varphi = F_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \gamma\, e_2 \otimes e_1$$

with the amount of shear $\gamma \in \mathbb{R}$.

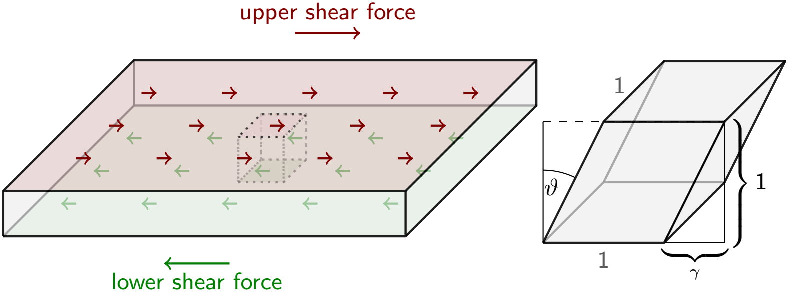


Pure shear stress

A **pure shear stress** is a stress tensor $T \in \mathrm{Sym}(3)$ of the form

$$T^s = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = s\,(e_1 \otimes e_2 + e_2 \otimes e_1)$$

with the amount of shear stress $s \in \mathbb{R}$.



Introduction

In isotropic nonlinear elasticity the Cauchy stress tensor is

$$\sigma = \beta_0 \mathbb{1} + \beta_1 B + \beta_{-1} B^{-1}$$

with $\beta_i = \beta_i\big(I_1(B), I_2(B), I_3(B)\big)$ and $B = FF^T$.

$$\text{Set} \qquad \sigma = T^s = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sigma = (\beta_0 + \beta_1 + \beta_{-1})\mathbb{1} + \begin{pmatrix} \beta_1 \gamma^2 & (\beta_1 - \beta_{-1})\gamma & 0 \\ (\beta_1 - \beta_{-1})\gamma & \beta_{-1}\gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies \gamma^2 (\beta_1 - \beta_{-1}) = 0 \qquad \text{then} \qquad \gamma = 0 \qquad \text{or} \qquad s = 0\,.$$

Pure shear Cauchy stress **never** corresponds to a simple shear deformation!

Questions:

- Independent of the elasticity law, which kind of deformations do correspond to pure shear Cauchy stress?
  [Destrade, Murphy, and Saccomandi 2012; Moon and Truesdell 1974; Mihai and Goriely 2011]
- Which of these deformations are suitable to be called 'shear'?
- Which constitutive requirements ensure that only 'shear' deformations correspond to pure shear Cauchy stress?

Which kind of deformations *correspond* to pure shear stress?

$B = FF^T$ and $\widehat{\sigma}(B)$ commute for **any** isotropic stress response.
$\iff$ B and $\widehat{\sigma}(B)$ are simultaneously diagonalizable.

$\widehat{\sigma}(B) = T^s$ can be diagonalized to $Q\,\mathrm{diag}(s, -s, 0)\,Q^T$ with

$$Q := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \in \mathrm{SO}(3)\,.$$

Thus $B = Q\,\mathrm{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2)\,Q^T$ with [Thiel, Voss, Martin, and Neff 2018a]

$$B = \frac{1}{2} \begin{pmatrix} \lambda_1^2 + \lambda_2^2 & \lambda_1^2 - \lambda_2^2 & 0 \\ \lambda_1^2 - \lambda_2^2 & \lambda_1^2 + \lambda_2^2 & 0 \\ 0 & 0 & 2\lambda_3^2 \end{pmatrix} \neq F_\gamma F_\gamma^T = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Which kind of deformations *correspond* to pure shear stress?

$$\widehat{\sigma}(B) = T^s = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Longleftrightarrow \quad B = \frac{1}{2} \begin{pmatrix} \lambda_1^2 + \lambda_2^2 & \lambda_1^2 - \lambda_2^2 & 0 \\ \lambda_1^2 - \lambda_2^2 & \lambda_1^2 + \lambda_2^2 & 0 \\ 0 & 0 & 2\lambda_3^2 \end{pmatrix}.$$

Then F is uniquely determined by triaxial stretch and simple shear

$$F = F_\gamma \, \mathrm{diag}(a, b, c)\, Q = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} Q$$

up to an arbitrary $Q \in \mathrm{SO}(3)$ with

$$a = \lambda_1 \lambda_2 \sqrt{\frac{2}{\lambda_1^2 + \lambda_2^2}}, \qquad b = \sqrt{\frac{\lambda_1^2 + \lambda_2^2}{2}}, \qquad c = \lambda_3, \qquad \gamma = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}.$$

Which of these deformations are *suitable* to be called shear?

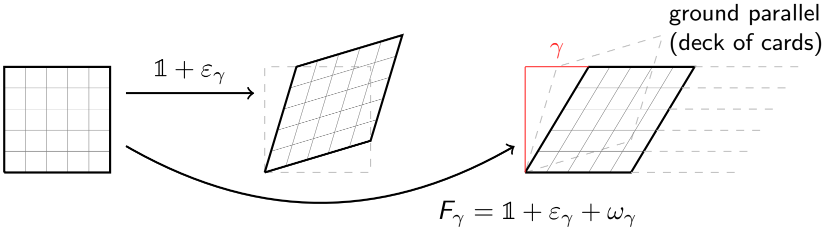
Linear Elasticity

The linear elastic Cauchy stress $\sigma_{\mathrm{lin}} = 2\mu \operatorname{dev} \varepsilon + \kappa \operatorname{tr} \varepsilon$ with $\varepsilon = \operatorname{sym}(F - \mathbb{1})$ and $\operatorname{dev} \varepsilon = \varepsilon - \frac{1}{3} \operatorname{tr} \varepsilon \mathbb{1}$ is a pure shear if and only if

$$F = \mathbb{1} + \underbrace{\begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\varepsilon \in \mathrm{Sym}(3)} + A\,, \qquad A \in \mathfrak{so}(3)\,.$$

$$F_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \underbrace{\begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\substack{\varepsilon_\gamma \in \mathrm{Sym}(3) \\ \text{infinitesimal pure} \\ \text{shear strain}}} + \underbrace{\begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ -\frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\substack{\omega_\gamma \in \mathfrak{so}(3) \\ \text{infinitesimal rotation}}}.$$

Which of these deformations are *suitable* to be called shear?



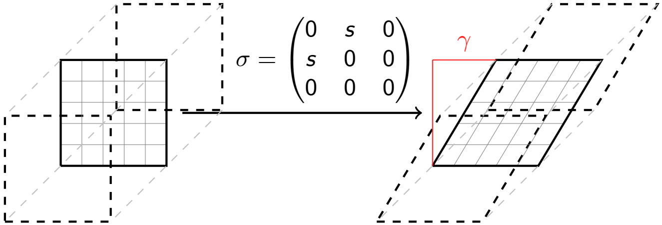
- The deformation F_γ is infinitesimally volume preserving, $\operatorname{tr} \varepsilon_\gamma = 0$.
- The deformation F_γ is planar, eigenvalue 1 to eigenvector e_3.
- The deformation F_γ is ground parallel, eigenvectors e_1 and e_3.

Which of these deformations are *suitable* to be called shear?

Generalizing from linear elasticity to nonlinear elasticity

- Pure shear Cauchy stress acts only in a plane
- Leonardo da Vinci: "Nessuno effetto è in natura sanza ragione" (No effect is in nature without cause) Codex Atlanticus

$\longrightarrow$ Nonlinear shear deformation should be planar



Which of these deformations are *suitable* to be called shear?

Definition: Finite shear deformation [Thiel, Voss, Martin, and Neff 2018b]

- The deformation F is volume preserving, $\det F = 1$.
- The deformation F is planar, eigenvalue 1 to eigenvector e_3.
- The deformation F is ground parallel, eigenvectors e_1 and e_3.

$\Longrightarrow$ there exists $\lambda \in \mathbb{R}_+$ with $\lambda_1 = \lambda$, $\lambda_2 = \frac{1}{\lambda}$ and $\lambda_3 = 1$.

$$\widehat{\sigma}(B) = T^s = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow V = \sqrt{B} = \frac{1}{2}\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 & 0 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & 0 \\ 0 & 0 & 2\lambda_3 \end{pmatrix}.$$

Which of these deformations are *suitable* to be called shear?

Finite pure shear stretch

$$V = \frac{1}{2}\begin{pmatrix} \lambda + \frac{1}{\lambda} & \lambda - \frac{1}{\lambda} & 0 \\ \lambda - \frac{1}{\lambda} & \lambda + \frac{1}{\lambda} & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{\alpha} + e^{-\alpha} & e^{\alpha} - e^{-\alpha} & 0 \\ e^{\alpha} - e^{-\alpha} & e^{\alpha} + e^{-\alpha} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

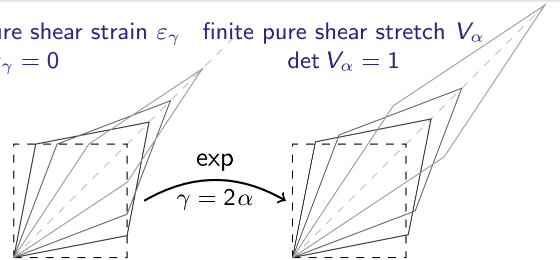
$$= \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 \\ \sinh(\alpha) & \cosh(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\text{matrix exponential}}{\exp} \underbrace{\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{infinitesimal pure shear strain}} =: V_\alpha \,, \quad \alpha := \log \lambda \,.$$

infinitesimal pure shear strain ε_γ

$\operatorname{tr} \varepsilon_\gamma = 0$

finite pure shear stretch V_α

$\det V_\alpha = 1$



Which of these deformations are *suitable* to be called shear?

$$\widehat{\sigma}(B) = T^s = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies F = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} Q.$$

with $\lambda_1 = \lambda$, $\lambda_2 = \frac{1}{\lambda}$, $\lambda_3 = 1$ and $\alpha = \log \lambda$:

Finite simple shear deformation

$$F = \begin{pmatrix} 1 & \tanh(2\alpha) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\cosh(2\alpha)}} & 0 & 0 \\ 0 & \sqrt{\cosh(2\alpha)} & 0 \\ 0 & 0 & 1 \end{pmatrix} Q$$

$$= \frac{1}{\sqrt{\cosh(2\alpha)}} \begin{pmatrix} 1 & \sinh(2\alpha) & 0 \\ 0 & \cosh(2\alpha) & 0 \\ 0 & 0 & \sqrt{\cosh(2\alpha)} \end{pmatrix} Q =: F_\alpha \,.$$

Which of these deformations are *suitable* to be called shear?

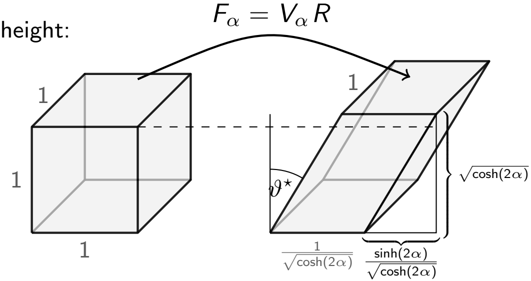
Finite simple shear deformation

A **finite simple shear deformation** is a mapping $\varphi\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the form

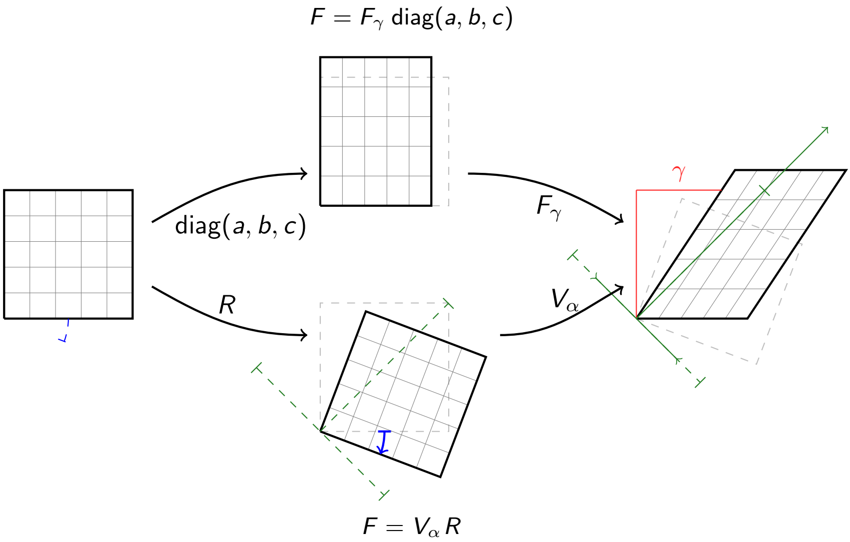
$$\nabla\varphi = F_\alpha = \frac{1}{\sqrt{\cosh(2\alpha)}}\begin{pmatrix} 1 & \sinh(2\alpha) & 0 \\ 0 & \cosh(2\alpha) & 0 \\ 0 & 0 & \sqrt{\cosh(2\alpha)} \end{pmatrix}$$

with the linearization $F_\alpha \overset{\alpha \ll 1}{\longrightarrow} F_\gamma$ and $\gamma = 2\alpha$.

increases the height:



Which of these deformations are *suitable* to be called shear?



Constitutive requirements in hyperelasticity

Which constitutive requirements ensure that only finite shear deformations correspond to pure shear Cauchy stress?

$$V_\alpha = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 \\ \sinh(\alpha) & \cosh(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q \cdot \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{pure shear deformation}} \cdot Q^T \quad \text{with } \lambda = e^\alpha,$$

$$\widehat{\sigma}(B) = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Q \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^T, \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\underbrace{\lambda_1 = \lambda\,, \quad \lambda_2 = \frac{1}{\lambda}\,, \quad \lambda_3 = 1}_{\text{singular values of } F} \qquad \Longrightarrow \qquad \underbrace{\sigma_1 = s\,, \quad \sigma_2 = -s\,, \quad \sigma_3 = 0}_{\text{principal Cauchy stresses}}\,.$$

Constitutive requirements in hyperelasticity

$$I_1 = \operatorname{tr} B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\,, \quad I_2 = \operatorname{tr}(\operatorname{Cof} B) = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2\,, \quad I_3 = \det B = \lambda_1^2\lambda_2^2\lambda_3^2$$

$$\sigma = \beta_0\,\mathbb{1} + \beta_1\,B + \beta_{-1}\,B^{-1} \quad \text{with } \beta_i = \beta_i\big(I_1(B), I_2(B), I_3(B)\big) \qquad \Longrightarrow$$

$$\beta_1 + \beta_{-1} = 0 \quad \text{and} \quad \beta_0 = 0 \qquad \forall\, \lambda \in \mathbb{R}_+ \text{ with } \lambda_1 = \lambda\,,\ \lambda_2 = \frac{1}{\lambda}\,,\ \lambda_3 = 1\,.$$

$$\beta_0 = \frac{2}{\sqrt{I_3}}\left(I_2\,\frac{\partial W}{\partial I_2} + I_3\,\frac{\partial W}{\partial I_3}\right), \qquad \beta_1 = \frac{2}{\sqrt{I_3}}\,\frac{\partial W}{\partial I_1}\,, \qquad \beta_{-1} = -2\sqrt{I_3}\,\frac{\partial W}{\partial I_2}\,,$$

$$\Longrightarrow\ \frac{\partial W}{\partial I_1} = \frac{\partial W}{\partial I_2} \quad \text{and} \quad I_2\,\frac{\partial W}{\partial I_2} + \frac{\partial W}{\partial I_3} = 0 \qquad \forall\ I_1 = I_2 \geq 3\,,\ I_3 = 1\,.$$

$$\sigma_i = \frac{\lambda_i}{\lambda_1\,\lambda_2\,\lambda_3}\,\frac{\partial W}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_2) \qquad \Longrightarrow$$

$$\lambda\,\frac{\partial W}{\partial \lambda_1} + \frac{1}{\lambda}\,\frac{\partial W}{\partial \lambda_2} = 0 \quad \text{and} \quad \frac{\partial W}{\partial \lambda_3} = 0 \qquad \forall\ \lambda_1 = \lambda\,,\ \lambda_2 = \frac{1}{\lambda}\,,\ \lambda_3 = 1\,.$$

Constitutive requirements in hyperelasticity

Tension-compression symmetry [Voss, Baaser, Martin, and Neff 2018]

Elastic energy $W\colon \mathrm{GL}^+(3) \to \mathbb{R}$ of the form

$$W(F) = W_{\mathrm{tc}}(F) + f(\det F)\,,$$

where W_{tc} is tension-compression symmetric, i.e. $W_{\mathrm{tc}}(F^{-1}) = W_{\mathrm{tc}}(F)$ and $f'(1) = 0$.

Hencky-type [Neff, Ghiba, and Lankeit 2015; Neff, Lankeit, Ghiba, Martin, and Steigmann 2015]

Elastic energy $W\colon \mathrm{GL}^+(3) \to \mathbb{R}$ of the form

$$W(F) = \psi\big(\|\operatorname{dev}\log V\|^2,\ |\operatorname{tr}\log V|^2\,\big)$$

for arbitrary functions $\psi\colon \mathbb{R}^2_+ \to \mathbb{R}$.

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References II

Thank you for your attention