Existence of Minimizers in Nonlinear Elastostatics of Micromorphic Solids

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Synonyms

Micromorphic; Microstructure; Polar materials; Solid mechanics; Structured continua; Variational methods

Overview

We consider the mathematical analysis of geometrically exact generalized continua of micromorphic type. The two-field minimization problem (for the macrodeformation field and the affine microdeformation field) is investigated in a variational form, namely, in the quasistatic, conservative load case. Two existence theorems in Sobolev spaces are given for the resulting nonlinear boundary value problems. These results comprise existence results for the micro-incompressible case and the Cosserat micropolar case. The mathematical analysis employs the direct methods of the calculus of variations and an extended Korn's inequality.

Introduction

This contribution addresses the mathematical analysis of geometrically exact generalized continua of micromorphic type. General continuum models involving independent rotations have been introduced by the Cosserat brothers [5] at the beginning of the last century. Since then, the Cosserat concept has been generalized in various directions; for an overview of these so-called microcontinuum theories, we refer to [3, 4, 7, 8, 19, 20]. The micromorphic model includes in a natural way size effects, i.e., small samples behave comparatively stiffer than large samples. These effects have recently received much attention in conjunction with nano-devices. From a computational point of view, theories with size effect are increasingly used to regularize non-well-posed situations, e.g., shear banding in elastoplasticity without hardening. It has already been shown that infinitesimal elastoplasticity augmented with purely elastic Cosserat effects indeed leads to a well-posed problem, for both the quasistatic and dynamic case [26, 27].

The mathematical analysis of general micromorphic solids restricted to the infinitesimal, linear elastic models is presented already, e.g., in [6, 9, 10] for linear micropolar models and in [11–13] for linear microstretch or micromorphic models. New developments regarding the weakest possible curvature contribution and invariance questions related to the infinitesimal model can be found in [14, 15, 30–33]. A connection of micromorphic models to gradient plasticity has been given in [35, 36].

The major difficulty of the mathematical treatment in the finite-strain case is related to the geometrically exact (fully frame indifferent) formulation of the theory and the appearance of nonlinear manifolds necessary for the description of the microstructure. In addition, coercivity turns out to be a delicate problem related to the possible fracture of the material. For related work on existence theorems in the case where coercivity is postulated from the beginning as a constitutive assumption, we refer the reader to [18, 39, 40].

This entry is organized as follows: first, we shortly review the basic concepts of the geometrically exact elastic micromorphic theories in a variational context, i.e., we formulate the quasistatic conservative load case as a two-field minimization problem. Then, the existence proof is presented. Since the two-field variational problem is only conditionally coercive, we need to introduce a modification for the applied loads in order to ensure first that the functional to be minimized is bounded below and second that the curvature contribution can be controlled. This modification of the loads, herein called principle of bounded external work, expresses nothing but the physical fact that by arbitrarily moving the solid in a force field, only a finite amount of work can be gained. Such a condition is, however, unnecessary in classical finite elasticity. With this preparation, existence of minimizers in Sobolev spaces is then established using the direct methods of the calculus of variations and an extended inequality of Korn type.

The application of this micromorphic model and some constitutive issues are discussed in [22, 24, 29]. The corresponding finite element implementation is treated, e.g., in [16, 17].

The Finite-Strain Elastic Micromorphic Model

Useful Notations

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with nonvanishing two-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$, we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times3}$ the set of real 3×3 second-order tensors, written with capital letters, and by $\mathfrak{T}(3)$ the set of all third-order tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = tr[XY^T]$, and thus, the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3$, $\mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $\mathbf{1}$, so that $tr[X] = \langle X, \mathbf{1}\rangle$. We let Sym and PSym denote the symmetric and positive-definite symmetric tensors, respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times3} \mid \det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid \det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^TX = \mathbf{1}\}$, and $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^TX = \mathbf{1},\ \det[X] = 1\}$ with corresponding Lie algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \mid X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times3} \mid tr[X] = 0\}$ of traceless tensors. We set $\mathrm{sym}(X) = \frac{1}{2}(X^T + X)$ and $\mathrm{skew}\,(X) = \frac{1}{2}(X - X^T)$ such that $X = \mathrm{sym}(X) + \mathrm{skew}(X)$. We write the classical polar decomposition in the form $F = R\,U = \mathrm{polar}(F)\,U$ with $R = \mathrm{polar}(F)$ the orthogonal part of F and U the positive-definite Biot stretch tensor. For a second-order tensor X, we define the third-order tensor $\mathfrak{h} = \mathrm{D}_x X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3}$. For third-order tensors $\mathfrak{h} \in \mathfrak{T}(3)$, we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\mathrm{sym}(\mathfrak{h}) := (\mathrm{sym}\mathfrak{h}^1, \mathrm{sym}\mathfrak{h}^2, \mathrm{sym}\mathfrak{h}^3)$ and $tr[\mathfrak{h}] := (tr[\mathfrak{h}^1], tr[\mathfrak{h}^2], tr[\mathfrak{h}^3]) \in \mathbb{R}^3$. Moreover, for any second-order tensor X, we define $X \cdot \mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h} \cdot X$ correspondingly. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$, we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times3})$ and we use ∇ in general only for column vectors in $\mathbb{R}^3$. We employ the standard notation of Sobolev spaces, i.e., $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_\circ(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega}\|X(x)\|$. We use capital letters to denote possibly large positive constants, e.g., C^+, K, and lowercase letters to denote possibly small positive constants, e.g., c^+, d^+.

Basic Equations

We now present the finite-strain micromorphic approach in a strictly Lagrangian description. We first introduce an independent kinematical field of microdeformations $P \in GL^+(3,\mathbb{R})$ together with its right polar decomposition

$$P = \overline{R}_p \cdot U_p = \mathrm{polar}(P) \cdot U_p = \overline{R}_p\, e^{\frac{\overline{\alpha}_p}{3}}\, \overline{U}_p, \quad \det[P] = e^{\overline{\alpha}_p},$$
$$\overline{U}_p = \frac{U_p}{\det[U_p]^{1/3}} \in \mathrm{SL}(3,\mathbb{R}),$$
$$\overline{P} = \frac{P}{\det[P]^{1/3}} \in \mathrm{SL}(3,\mathbb{R}) \tag{1}$$

with $\overline{R}_p \in \mathrm{SO}(3,\mathbb{R})$ and $\overline{U_p} \in \mathrm{PSym}(3,\mathbb{R}) \cap \mathrm{SL}(3,\mathbb{R})$. The microdeformations P are meant to describe the substructure of the material which can rotate, stretch, shear, and shrink. We refer to $\overline{R}_p$ as microrotations. Following Eringen [7, p. 13], we distinguish the general micromorphic case, $P \in GL^+(3,\mathbb{R}) = \mathbb{R}^+ \cdot \mathrm{SL}(3,\mathbb{R})$ with

9 additional degrees of freedom (DOF); the micro-incompressible micromorphic case, $P \in \mathrm{SL}(3,\mathbb{R})$ with 8 DOF; the microstretch case, $P \in \mathbb{R}^+ \cdot \mathrm{SO}(3,\mathbb{R})$ with 4 DOF; and the micropolar case, $P \in \mathrm{SO}(3,\mathbb{R})$ with only 3 additional DOF. The theory with voids is included if $P \in \mathbb{R}^+ \cdot \mathbf{1}$ with one DOF.

The micromorphic theory we deal with can formally be obtained by introducing the multiplicative decomposition of the macroscopic deformation gradient F into independent microdeformation P and the micromorphic, nonsymmetric right stretch tensor $\overline{U}$ (first Cosserat deformation tensor, the relative distortion) with

$$F = P \cdot \overline{U}, \quad \overline{U} = P^{-1}F, \quad \overline{U} \in GL^+(3,\mathbb{R}) \tag{2}$$

leading altogether to a micro-compressible, micromorphic formulation.

In the quasistatic case, the micromorphic theory is derived from a two-field variational principle by postulating the following "Euclidean action" [5, p. 156] I for the finite macroscopic deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ and the independent microdeformation $P : [0,T] \times \overline{\Omega} \mapsto GL^+(3,\mathbb{R})$:

$$\begin{aligned} I(\varphi,P) = &\int_\Omega W(F,P,\mathrm{D_x}P) - \Pi_f(\varphi) - \Pi_M(P)\,\mathrm{dV} \\ &- \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS} \\ &- \int_{\Gamma_C} \Pi_{M_c}(P)\,\mathrm{dS} \mapsto \min.\ \text{w.r.t. } (\varphi,P), \\ P_{|_\Gamma} = P_\mathrm{d}, &\quad \varphi_{|_\Gamma} = g_\mathrm{d}(t) \end{aligned} \tag{3}$$

The elastically stored energy density W depends on the macroscopic deformation gradient $F = \nabla\varphi$ as usual but in addition on the microdeformation P together with their first-order space derivatives, represented through the third-order tensor $\mathrm{D_x}P$. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions g_d,P_d for displacements and microdeformations, respectively, can be prescribed, while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. The potential of external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary, where the potential of applied surface couples Π_{M_c} is applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$, corresponding natural boundary conditions for φ and P apply, which are obtained automatically in the variational process.

Variation of the action I with respect to φ yields the traditional equation for balance of linear momentum, and variation of I with respect to P yields the additional balance of moment of momentum [see 29]. The standard conclusion from frame-indifference (invariance of the free energy under superposed rigid body motions) is as follows: for all

$$\begin{aligned} \forall Q \in \mathrm{SO}(3,\mathbb{R}) \Rightarrow W(F, P, \mathrm{D_x}P) &= W(QF, QP, \mathrm{D_x}[QP]) \\ &= W(QF, QP, Q\mathrm{D_x}P) \end{aligned}$$

and this leads to the reduced representation of the energy (specify $Q = \overline{R}_p^T$):

$$\begin{aligned} W(F,\overline{P},\mathrm{D_x}P) &= W(\overline{R}_p^T F, \overline{R}_p^T P, \overline{R}_p^T \mathrm{D_x}P) \\ &= W(U_p\overline{U}, U_p, \overline{R}_p^T \mathrm{D_x}P) \\ &= W^\sharp(\overline{U}, U_p, \mathfrak{K}_p, \nabla\overline{\alpha}_p) \end{aligned} \tag{4}$$

where for $\overline{P} = \overline{R}_p\,\overline{U}_p \in \mathrm{SL}(3,\mathbb{R})$ we set

$$\begin{aligned} \mathfrak{K}_p &:= \overline{R}_p^T \mathrm{D_x}\overline{P} \\ &= \left(\overline{R}_p^T \nabla(\overline{P}.e_1), \overline{R}_p^T \nabla(\overline{P}.e_2), \overline{R}_p^T \nabla(\overline{P}.e_3)\right) \\ &\in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \end{aligned} \tag{5}$$

For a geometrically exact (macroscopically isotropic) theory, we assume in the following an additive split of the total free-energy density into micromorphic relative local stretch (macroscopic), stretch of the substructure itself (microscopic), and micromorphic curvature part according to

$$W^{\sharp} = \underbrace{W_{\text{mp}}(\overline{U})}_{\text{relative macroscopic energy}} + \underbrace{W_{\text{foam}}(\overline{U}_p, \overline{\alpha}_p)}_{\text{microscopic local energy}} + \underbrace{W_{\text{curv}}(K_p, \nabla\overline{\alpha}_p)}_{\text{microscopic interaction energy}} \tag{6}$$

since a possible coupling between $\overline{U}$ and $\mathfrak{K}_p$ for centrosymmetric bodies can be ruled out [37, p. 14].

The Elastic Macroscopic Micromorphic Strain Energy Density

For a macroscopic theory which is relevant mainly for small elastic strain, we require that $W_{\text{mp}}(\overline{U})$ is a nonnegative isotropic quadratic form (leading to a physically linear problem). This covers already many cases of physical interest. For the local energy contribution elastically stored in the substructure, we assume the nonlinear expression

$$\begin{aligned} W_{\text{foam}}(U_p) &= \underbrace{\mu^{\text{m}}\|\frac{U_p}{det[U_p]^{(1/3)}} - \mathbf{1}\|^2}_{\text{isochoric substructure energy}} \\ &\quad \underbrace{+\frac{\lambda^{\text{m}}}{4}\left((det[U_p]-1)^2 + (\frac{1}{det[U_p]}-1)^2\right)}_{\text{volumetric energy}} \\ &= \mu^{\text{m}}\|\overline{U}_p - \mathbf{1}\|^2 \\ &\quad + \frac{\lambda^{\text{m}}}{4}\left((e^{\overline{\alpha}_p}-1)^2 + (e^{-\overline{\alpha}_p}-1)^2\right) \\ &=: W_{\text{foam}}(\overline{U}_p, \overline{\alpha}_p) \end{aligned} \tag{7}$$

avoiding self-interpenetration in a variational setting, since $W_{\text{foam}} \to \infty$ as $det[P] = det[U_p] \to 0$ if $\lambda^{\text{m}} > 0$.

The most general isotropic quadratic form of W_{mp} is

$$\begin{aligned} W_{\text{mp}}(\overline{U}) &= \mu_e\|\,\text{sym}(\overline{U}-\mathbf{1})\|^2 + \mu_c\|\,\text{skew}(\overline{U}-\mathbf{1})\|^2 \\ &\quad + \frac{\lambda_e}{2}\,\text{tr}\left[\text{sym}(\overline{U}-\mathbf{1})\right]^2 \end{aligned} \tag{8}$$

with material constants μ_e, μ_c, λ_e such that $\mu_e,\ 3\lambda_e + 2\mu_e, \mu_c \geq 0$ from non-negativity of (8) [see 7].

The coefficients μ_e, λ_e are effective elastic constants which in general do not coincide with the classical Lamé constants. The so-called Cosserat couple modulus μ_c (rotational couple modulus) remains for the moment unspecified, but we note that $\mu_c = 0$ is physically possible, even in the micropolar case, since the micromorphic reaction stress $D_{\overline{U}}W_{\text{mp}}(\overline{U}) \cdot \overline{U}^T$ is not symmetric in general, i.e., the problem does not decouple [28]. By formal similarity with the classical formulation, we may call μ^{m}, λ^{m} the microscopic Lamé moduli of the affine substructure.

The Nonlinear Elastic Curvature Energy Density

The curvature energy is responsible for the size-dependent resistance of the substructure against local twisting and inhomogeneous volume change. Thus, inhomogeneous microstructural rearrangements are penalized. For the curvature term, to be specific, we assume

$$\begin{aligned} W_{\text{curv}}(\mathfrak{K}_p, \nabla\overline{\alpha}_p) &= \mu\frac{L_c^{1+p}}{12}\left(1+\alpha_4 L_c^q\,\|\mathfrak{K}_p\|^q\right) \\ &\times \left(\alpha_5\,\|\text{sym}\mathfrak{K}_p\|^2 + \alpha_6\|\,\text{skew}\mathfrak{K}_p\|^2 + \alpha_7\,tr\left[\mathfrak{K}_p\right]^2\right)^{\frac{1+p}{2}} \\ &+ \mu\frac{L_c^{1+p}}{12}\left(\alpha_8\,\|\nabla\overline{\alpha}_p\|^{1+p} + \alpha_8 L_c\|\,\nabla\overline{\alpha}_p\,\|^{2+p}\right) \end{aligned} \tag{9}$$

where $L_c>0$ is setting an internal length scale with units of length. It is to be noted that we have decoupled the curvature coming from inhomogeneous volume changes and from pure twisting. The values $\alpha_4 \geq 0, p > 0$ and $q \geq 0$ are additional material constants. We mean $tr\left[\mathfrak{K}_p\right]^2 = \|\,tr\left[\mathfrak{K}_p\right]\|^2$ by abuse of notation.

In the finite-strain regime, W_{curv} should preferably be coercive in the sense that we impose pointwise

$$\exists c^+ > 0 \, \exists r > 1 : \; \forall \mathfrak{K}_p \in \mathfrak{T}(3) \; \forall \xi \in \mathbb{R}^3 : \\ W_{\text{curv}}(\mathfrak{K}_p, \xi) \geq c^+ \, ||(\mathfrak{K}_p, \xi)||^r \tag{10}$$

or less demanding

$$\exists r > 1 : \quad \frac{W_{\text{curv}}(\mathfrak{K}_p, \xi)}{||(\mathfrak{K}_p, \xi)^r||} \to \infty \quad \text{as } ||(\mathfrak{K}_p, \xi)|| \to \infty \tag{11}$$

which implies necessarily $\alpha_6, \alpha_8 > 0$ in (9). Observe that our formulation of the micromorphic curvature tensor is mathematically convenient in the sense that $||\mathfrak{K}_p|| = ||\overline{R}_p^T \, \mathrm{D_x}\overline{P}|| = ||\mathrm{D_x}\overline{P}||$ provides pointwise control of all first derivatives of $\overline{P}$ independent of the values of $\overline{P}$ itself.

Note that the presented formulation includes a finite-strain Cosserat micropolar model as a special case, if we set $\overline{P} = \overline{R} \in \mathrm{SO}(3, \mathbb{R})$. In the Cosserat case, an alternative curvature tensor based on $\overline{R}^T \, \mathrm{Curl}\overline{R}$ suggests itself [34].

Altogether, we have the following correspondence of limit problems:

$$\begin{aligned} \lambda^{\mathrm{m}} \to \infty \quad &\Rightarrow \quad \text{micro-incompressible model : manifold } \mathrm{SL}(3,\mathbb{R}) \\ \mu^{\mathrm{m}} \to \infty \quad &\Rightarrow \quad \text{microstretch model : manifold } \mathbb{R}^+ \cdot \mathrm{SO}(3,\mathbb{R}) \\ \mu^{\mathrm{m}}, \lambda^{\mathrm{m}} \to \infty \quad &\Rightarrow \quad \text{micropolar model : manifold } \mathrm{SO}(3,\mathbb{R}) \\ \mu^{\mathrm{m}}, \lambda^{\mathrm{m}}, \mu_c \to \infty \quad &\Rightarrow \quad \text{higher (second) gradient continua of Mindlin-type} \end{aligned} \tag{12}$$

Analysis of the Mathematical Problem

The Micromorphic Problem in Variational Form

Let us gather the obtained three-field problem posed in a variational form. The task is to find a triple $(\varphi, \overline{P}, \overline{\alpha}_p) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathrm{SL}(3, \mathbb{R}) \times \mathbb{R}$ of macroscopic deformation φ and independent microdeformation $P = e^{\frac{\overline{\alpha}_p}{3}} \overline{P}$, minimizing the energy functional I with

$$\begin{aligned} I(\varphi, \overline{P}, \overline{\alpha}_p) = \int_\Omega &[W_{\text{mp}}(P^{-1}\nabla\varphi) + W_{\text{foam}}(\overline{U}_p, \overline{\alpha}_p) \\ &+ W_{\text{curv}}(\overline{R}_p^T \, \mathrm{D_x}\overline{P}, \nabla \overline{\alpha}_p) - \Pi_f(\varphi) \\ &- \Pi_M(P)]\mathrm{dV} - \int_{\Gamma_S} \Pi_N(\varphi) \, \mathrm{dS} \\ &- \int_{\Gamma_C} \Pi_{M_c}(P) \, \mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{P}, \overline{\alpha}_p) \end{aligned} \tag{13}$$

under the constraints

$$\begin{aligned} \overline{U}_p = \overline{R}_p^T \, \overline{P}, \quad \overline{R}_p = \mathrm{polar}(\overline{P}) \\ \overline{U} = P^{-1}\nabla\varphi, \quad P = e^{\frac{\overline{\alpha}_p}{3}} \overline{P} \end{aligned} \tag{14}$$

and the Dirichlet boundary conditions

$$\begin{aligned} &\varphi_{|_\Gamma} = g_{\mathrm{d}}, \quad \overline{R}_{p|_\Gamma} = \overline{R}_{p\mathrm{d}}, \quad \overline{U}_{p|_\Gamma} = \overline{U}_{p\mathrm{d}} \Rightarrow \\ &\overline{P}_{|_\Gamma} = \overline{R}_{p\mathrm{d}} \, \overline{U}_{p\mathrm{d}}, \overline{\alpha}_{p|_\Gamma} = \overline{\alpha}_{p\mathrm{d}} \end{aligned} \tag{15}$$

Here, the constitutive assumptions on the densities are taken to be

$$\begin{aligned} W_{\text{mp}}(\overline{U}) &= \mu_e || \, \mathrm{sym}(\overline{U} - \mathbf{1})||^2 \\ &+ \mu_c \, ||\mathrm{skew}(\overline{U})||^2 + \frac{\lambda_e}{2} \, tr\left[\mathrm{sym}(\overline{U} - \mathbf{1})\right]^2 \\ W_{\text{foam}}(\overline{U}_p, \overline{\alpha}_p) &= \mu^{\mathrm{m}} || \, \overline{U}_p - \mathbf{1}||^2 \\ &+ \frac{\lambda^{\mathrm{m}}}{4} \left((e^{\overline{\alpha}_p} - 1)^2 + (e^{-\overline{\alpha}_p} - 1)^2 \right) \\ W_{\text{curv}}(\mathfrak{K}_p, \nabla \overline{\alpha}_p) &= \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 L_c^q \, ||\mathfrak{K}_p||^q\right) \\ &\times \left(\alpha_5 || \, \mathrm{sym}\mathfrak{K}_p||^2 + \alpha_6 || \, \mathrm{skew}\mathfrak{K}_p||^2 + \alpha_7 \, tr\left[\mathfrak{K}_p\right]^2\right)^{\frac{1+p}{2}} \\ &+ \mu \frac{L_c^{1+p}}{12} \left(\alpha_8 \, ||\nabla \overline{\alpha}_p||^{1+p} + \alpha_8 L_c || \, \nabla \overline{\alpha}_p \, ||^{2+p}\right) \\ \mathfrak{K}_p &= \overline{R}_p^T \, \mathrm{D_x}\overline{P} = \left(\overline{R}_p^T \nabla(\overline{P}.e_1), \overline{R}_p^T \nabla(\overline{P}.e_2), \overline{R}_p^T \nabla(\overline{P}.e_3)\right) \end{aligned} \tag{16}$$

It is assumed that $\mu_e,\ \lambda_e > 0,\ \mu_c \geq 0$, and $\mu^{\mathrm{m}},\ \lambda^{\mathrm{m}}, L_c > 0$. The parameters $\alpha_i,\ i = 1, .., 8$ are dimensionless weighting factors. If not stated otherwise, we assume that $\alpha_5 > 0, \alpha_6 > 0, \alpha_8 > 0, \alpha_7 \geq 0$.

Traditionally, in the conservative dead load case, one would have

$$\begin{array}{ll} \Pi_f(\varphi) = \langle f, \varphi \rangle, & \Pi_M(P) = \langle M, P \rangle \\ \Pi_N(\varphi) = \langle N, \varphi \rangle, & \Pi_{M_c}(P) = \langle M_c, P \rangle \end{array} \tag{17}$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3\times3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3\times3})$.

For the treatment of our model, we need to assume, however, that the external potentials, describing the configuration dependent applied loads, are continuous with respect to the topology of $L^1(\Omega), L^1(\Gamma_S), L^1(\Gamma_C)$, respectively, and satisfy in addition the crucial condition

$$\begin{aligned} &\exists\, C^+ > 0 \quad \forall \varphi \in L^1(\Omega, \mathbb{R}^3), \\ &P \in L^1(\Omega, GL^+(3, \mathbb{R}))\ : \\ &\int_\Omega \Pi_f(\varphi) - \Pi_M(P)\, \mathrm{dV}, \\ &\int_{\Gamma_S} \Pi_N(\varphi)\, \mathrm{dS}, \\ &\int_{\Gamma_C} \Pi_{M_c}(P)\, \mathrm{dS} \leq C^+ \end{aligned} \tag{18}$$

While continuity is satisfied, e.g., for the dead load case $\Pi_f(\varphi) = \langle f, \varphi \rangle$ and $f \in L^\infty(\Omega)$, the second condition (18) restricts attention to "bounded external work." If we want to describe a situation corresponding to the classical dead load case, we could take

$$\Pi_f(\varphi) = \frac{1}{1 + [\|\varphi(x)\| - K^+]_+} \langle f(x), \varphi(x) \rangle \tag{19}$$

for some large positive constant K^+ and $[\cdot]_+$ the positive part of a scalar argument. It suffices now that $f \in L^1(\Omega)$, then $\int_\Omega \Pi_f(\varphi)\, \mathrm{dV} \leq C^+$, independent of $\varphi \in L^1(\Omega)$.

The new condition (18) can be rephrased as saying that only a finite amount of work can be performed against the external loads, regardless of the magnitude of translation and microdeformation. This is certainly true for any real field of applied loads [23]. The mathematical consequence is that when considering infimizing sequences without this assumption, it could happen that the curvature contribution is not controlled while the total energy remains bounded.

The Coercivity Inequality

We distinguish three different situations:

Case 1: $\mu_c > 0,\ \alpha_4 \geq 0,\ p \geq 1,\ q \geq 0$. Elastic macro-stability, local first-order micromorphic. Fracture excluded

Case 2: $\mu_c = 0,\ \alpha_4 > 0,\ p \geq 1,\ q > 1$. Elastic pre-stability, nonlocal second-order micromorphic, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Fracture excluded for bounded external work

Case 3: $\mu_c = 0,\ \alpha_4 = 0,\ 0 < p \leq 1,\ q = 0$. Elastic pre-stability, nonlocal second-order micromorphic theory, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $\varphi \notin W^{1,1}(\Omega, \mathbb{R}^3)$, due to lack of elastic coercivity, including fracture in multiaxial situations.

We refer to $0 < p < 1,\ q \geq 0$ as the subcritical case, to $p = 1,\ q \geq 0$ as the critical case, and to $p \geq 1,\ q > 1$ as the supercritical case. We will treat the first two cases mathematically.

The decisive analytical tool underlying the treatment of case 2 (supercritical, $\mu_c = 0$) is the following inequality establishing coercivity:

Theorem 1. (Extended Korn's First Inequality). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with nonvanishing 2-dimensional Hausdorff measure. Define $H^{1,2}_\circ(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega)\ |\phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^0(\overline{\Omega}, \mathrm{GL}(3, \mathbb{R}))$. Then*

$$\exists\, c^+ > 0\ \forall\, \phi \in H_\circ^{1,2}(\Omega,\Gamma):$$
$$\left\| \nabla\phi F_p^{-1}(x) + F_p^{-T}(x)\nabla\phi^T \right\|^2_{L^2(\Omega)}$$
$$\geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}$$

Proof. The proof of this version of Korn's inequality is presented in [38], which is improving on a similar result of the present author [21] where the possible validity of the inequality has been first observed.

Existence Results for the Geometrically Exact Elastic Micromorphic Model

The following results have first been obtained in [23].

Theorem 2. (Existence for Elastic Micromorphic Model: Case 1). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega,\mathbb{R}^3)$ and $P_{\mathrm{d}} \in W^{1,1+p}(\Omega, GL^+(3,\mathbb{R}))$. Moreover, let the applied external potentials satisfy (18). Then (13) with material constants conforming to case 1 and $p > 1$ admits at least one minimizing solution triple $(\varphi, \overline{P}, \overline{\alpha}_p) \in H^1(\Omega,\mathbb{R}^3)\times W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R}))\times W^{1,2+p}(\Omega,\mathbb{R})$.*

Proof. The proof is based on the direct methods of the calculus of variations. The influence of the external potentials is gathered in writing $\Pi(\varphi,P)$. With the prescription of $(g_{\mathrm{d}}, P_{\mathrm{d}})$ as data of the problem, it is clear that $I < \infty$ for exactly this pair of functions after decomposing P_{d} in its rotational, isochoric stretch and volumetric stretch. Since (18) is assumed, it is also clear that I is bounded below for all $\varphi \in L^2(\Omega,\mathbb{R}^3)$ and $P \in L^2(\Omega, GL^+(3,\mathbb{R}))$.

We may therefore choose infimizing "sequences of triples"

$$(\varphi^k, \overline{P}^k, \overline{\alpha}_p^k) \in H^1(\Omega,\mathbb{R}^3)\times W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R})) \times W^{1,2+p}(\Omega,\mathbb{R}) \qquad (20)$$

such that

$$\lim_{k\to\infty} I(\varphi^k, \overline{P}^k, \overline{\alpha}_p^k) = \inf\{I(\varphi,\overline{P},\overline{\alpha}_p)\,|\,\varphi\in L^1(\Omega,\mathbb{R}^3),\ \overline{P}\in L^1(\Omega,\mathrm{SL}(3,\mathbb{R})),\ \overline{\alpha}_p\in L^1(\Omega,\mathbb{R})\} \qquad (21)$$

The total curvature contribution W_{curv} along this sequence is bounded independent of the number k again on account of (18).

Observe now that the micromorphic curvature term $\mathfrak{K}_p$ controls $\overline{P} \in W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R}))$, in view of $\|\mathfrak{K}_p\| = \left\|\overline{R}_p^T\,\mathrm{D_x}\overline{P}\right\| = \|\mathrm{D_x}\overline{P}\|$ pointwise, the assumption that $\alpha_5, \alpha_6 > 0$ and the application of Poincaré's inequality with the Dirichlet conditions on $\overline{P}$. Moreover, since $\alpha_8 > 0$ we obtain boundedness of $\overline{\alpha}_p^k \in W^{1,2+p}(\Omega,\mathbb{R})$, again independent of $k\in\mathbb{N}$. This result remains true already without specification of Dirichlet boundary conditions for $\overline{\alpha}_p$ since the term $e^{\overline{\alpha}_p}$ estimates any L^q-norm of $\overline{\alpha}_p$. For $p > 1$ Sobolev's embedding shows that we can choose a subsequence, not relabeled, such that strongly

$$\overline{\alpha}_p^k \to \widehat{\overline{\alpha}}_p \in C^0(\Omega,\mathbb{R}), \quad k\to\infty \qquad (22)$$

Now we may extract a subsequence again denoted by $\overline{P}^k$ converging strongly in $L^{1+p}(\Omega)$ to an element $\widehat{P} \in W^{1,1+p}(\Omega,\mathbb{M}^{3\times 3})$ since $p > 0$ by assumption. Moreover, a further subsequence can be found, such that the curvature tensor $\mathfrak{K}_{p,k}$ converges weakly to some $\widehat{\mathfrak{K}_p}$ in $L^{1+p}(\Omega)$. For $1 < (1+p) < 3$, the embedding

$$W^{1,1+p}(\Omega) \subset L^{\frac{3(1+p)}{3-(1+p)}-\delta}(\Omega), \quad \delta \geq 0 \qquad (23)$$

for three space dimensions is compact for $\delta > 0$ and shows that the subsequence $\overline{P}^k$ can be chosen such that it converges indeed strongly in the topology of $L^{6-\delta}(\Omega)$, since we have moreover $p \geq 1$, which implies immediately that $\widehat{P} \in W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R}))$. If $1+p \geq 3$, we can use better embeddings to have the same conclusion.

Because $\mu_c > 0$, we have the simple algebraic estimate

$$
\begin{aligned}
W_{\text{mp}}(P^{-1,k}F^k) &\geq \mu_c \, \|P^{-1,k}F^k - \mathbf{1}\|^2 \\
&= \mu_c \left(\|P^{-1,k}F^k\|^2 - 2\langle P^{-1,k}F^k, \mathbf{1}\rangle + 3 \right) \\
&\geq \mu_c \left(\|\overline{U}_k\|^2 - 2\sqrt{3}\|\overline{U}_k\| + 3 \right)
\end{aligned} \tag{24}
$$

implying the boundedness of the micromorphic stretch $\overline{U}_k = P^{-1,k}F^k$ in $L^2(\Omega)$. Moreover, by Hölder's inequality, we obtain

$$
\begin{aligned}
\|F^k\|_{s,\Omega} &= \|P^k P^{-1,k} F^k\|_{s,\Omega} \\
&\leq \|P^k\|_{r_1,\Omega} \| P^{-1,k}F^k\|_{r_2,\Omega} \\
&= ||e^{\frac{\overline{\alpha}_p^k}{3}} \, \overline{P}^k \, ||_{r_1,\Omega} \, ||P^{-1,k}F^k||_{r_2,\Omega} \\
&\leq \sup_{x\in\Omega} e^{\frac{\overline{\alpha}_p^k(x)}{3}} || \, \overline{P}^k \, ||_{r_1,\Omega} \, ||P^{-1,k}F^k||_{r_2,\Omega}, \\
\frac{1}{s} &= \frac{1}{r_1} + \frac{1}{r_2}
\end{aligned} \tag{25}
$$

Since $\overline{P}^k$ is bounded in $L^6(\Omega)$ (see (23)) and $P^{-1,k}F^k$ is bounded in $L^2(\Omega)$ and $\overline{\alpha}_p^k$ is strongly converging in $C^0(\Omega, \mathbb{R})$ (22), we may choose $r_1 = 6$, $r_2 = 2$ to obtain boundedness of $F^k = \nabla\varphi_k$ in $L^s(\Omega)$, $s = \frac{3}{2}$. Using the Dirichlet boundary conditions for φ_k and the generalized Poincaré inequality, we get

$$
|| \, \varphi_k \, ||_{W^{1,s}(\Omega,\mathbb{R}^3)} \leq \text{Const} \tag{26}
$$

By the boundedness of φ^k in $W^{1,s}(\Omega, \mathbb{R}^3)$ we may extract a subsequence, not relabeled, such that $\varphi^k \rightharpoonup \hat{\varphi} \in W^{1,s}(\Omega, \mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of (φ^k, P^k) such that $\overline{U}_k = P^{-1,k}F^k$ converges weakly in $L^2(\Omega)$ to some element $\widehat{\overline{U}}$ on account of the boundedness of the stretch energy and $\mu_c > 0$.

We have already shown that for $p \geq 1$, the sequence $\overline{P}^k$ converges indeed strongly in $L^r(\Omega)$ to an element $\widehat{\overline{P}} \in W^{1,1+p}(\Omega, \text{SL}(3, \mathbb{R}))$. Therefore,

$$
\begin{aligned}
\overline{P}^{-1,k} &= \frac{1}{\det[\overline{P}^k]} \text{Adj}\,\overline{P}^k \to \frac{1}{\det[\widehat{\overline{P}}]} \text{Adj}\widehat{\overline{P}} \\
&= \widehat{\overline{P}}^{-1} \quad \text{in}\, L^{\frac{r}{2}}(\Omega, \text{SL}(3, \mathbb{R})), \\
r &= \frac{3(1+p)}{(3-(1+p))} - \delta, \quad \text{if } 1 < (1+p) < 3
\end{aligned} \tag{27}
$$

and we obtain for $p > 1$ that $\overline{P}^{-1,k} \to \widehat{\overline{P}}^{-1}$ strongly in $L^{3+\tilde{\delta}}(\Omega, \text{SL}(3, \mathbb{R}))$, $\tilde{\delta} > 0$. Moreover,

$$
P^{-1,k} = e^{-\frac{\overline{\alpha}_p^k}{3}} \, \overline{P}^{-1,k} \to \widehat{P}^{-1} = e^{-\frac{\widehat{\overline{\alpha}}_p}{3}} \, \widehat{\overline{P}}^{-1,k} \tag{28}
$$

on account of the strong convergence of $\overline{\alpha}_p^k$. Thus, $\overline{P}^{-1,k} F^k$ converges certainly weakly to $\widehat{P}^{-1} F$ in $L^1(\Omega)$ on account of Hölder's inequality (sharp). The weak limit in $L^1(\Omega)$ must coincide with the weak limit of $\overline{U}_k$ in $L^2(\Omega)$. Hence, the identity $\widehat{\overline{U}} = \widehat{P}^{-1} \nabla\hat{\varphi}$ holds.

Since the mapping polar $GL^+(3, \mathbb{R}) \mapsto \text{SO}(3, \mathbb{R})$ is a bounded continuous function on invertible matrices with positive determinant, it generates a nonlinear superposition operator

$$
\text{polar}(\cdot) : L^r(\Omega, GL^+(3, \mathbb{R})) \mapsto L^r(\Omega, \text{SO}(3, \mathbb{R})) \tag{29}
$$

which, moreover, is continuous [1, p. 101, Th. 3.7]. Thus, $\overline{R}_k = \text{polar}(\overline{P}_k) \to \widehat{\overline{R}} = \text{polar}(\widehat{\overline{P}})$ strongly in $L^r(\Omega)$, and a similar argument as for the sequence $\overline{U}_k$ shows that

$$
\mathfrak{K}_{p,k} \rightharpoonup \widehat{\mathfrak{K}_p} = \text{polar}(\widehat{P})^T \text{D}_x \widehat{P} \tag{30}
$$

in $L^{1+p}(\Omega)$, weakly. Again on account of $\overline{P}^k \to \widehat{\overline{P}}$ in $L^r(\Omega, \text{SL}(3, \mathbb{R}))$, we infer now

$$
\overline{U}_p^k = \sqrt{\overline{P}^{k,T} \, \overline{P}^k} \to \sqrt{\widehat{\overline{P}}^T \widehat{\overline{P}}} = \widehat{\overline{U}}_p \quad \text{in } L^r(\Omega, \text{SL}(3, \mathbb{R})) \tag{31}
$$

because the map $\mathbb{M}^{3\times3}\mapsto\mathrm{PSym}(3)$, $X\mapsto\sqrt{X^TX}$ is continuous and has linear growth.

Since the total energy is convex in the extended set of variables $(\overline{U},\overline{U}_p,\mathfrak{K}_p,\nabla\overline{\alpha}_p)$ and continuous w.r.t. $\overline{\alpha}_p$, and the external potential Π is continuous w.r.t. strong convergence in $L^1(\Omega)$ on account of (18), we get

$$\begin{aligned}I(\hat{\varphi},\widehat{\overline{P}},\widehat{\overline{\alpha}}_p) &= \int_\Omega W_{\mathrm{mp}}(\widehat{\overline{U}})+W_{\mathrm{foam}}(\widehat{\overline{U}}_p,\widehat{\overline{\alpha}}_p)\\ &+W_{\mathrm{curv}}(\widehat{\mathfrak{K}_p},\nabla\widehat{\overline{\alpha}}_p)\,\mathrm{dV}-\Pi(\hat{\varphi},\widehat{P})\\ &\le \liminf_{k\to\infty}\int_\Omega W_{\mathrm{mp}}(\overline{U}_k)+W_{\mathrm{foam}}(\overline{U}_p^k)\\ &+W_{\mathrm{curv}}(\mathfrak{K}_{p,k},\nabla\overline{\alpha}_p^k)\,\mathrm{dV}-\Pi(\varphi_k,P_k)\\ &=\lim_{k\to\infty}I(\varphi^k,\overline{P}^k,\overline{\alpha}_p^k)\\ &=\inf\{I(\varphi,\overline{P},\overline{\alpha}_p)\,|\,\varphi\in L^1(\Omega,\mathbb{R}^3),\\ &\qquad\overline{P}\in L^1(\Omega,\mathrm{SL}(3,\mathbb{R})),\ \overline{\alpha}_p\in L^1(\Omega,\mathbb{R})\}\end{aligned}\tag{32}$$

which implies that the limit triple $(\hat{\varphi},\widehat{P},\widehat{\overline{\alpha}}_p)$ is a minimizer. Note that the limit microdeformations $P=e^{\frac{\overline{\alpha}_p}{3}}\,\overline{R}_p\,\overline{U}_p$ may fail to be continuous, if $p\le2$ (nonexistence or limit case of Sobolev embedding). Moreover, uniqueness cannot be ascertained, since $\mathrm{SL}(3,\mathbb{R})$ is a nonlinear manifold (and the considered problem is indeed highly nonlinear), such that convex combinations in $\mathrm{SL}(3,\mathbb{R})$ may leave $\mathrm{SL}(3,\mathbb{R})$. Since the functional I is differentiable, the minimizing pair is a stationary point and therefore a solution of the corresponding field equations. Note again that the limit microdeformations may fail to be continuously distributed in space. That under these unfavorable circumstances a minimizing solution may nevertheless be found is entirely due to $\mu_c>0$ and $p>1$. The proof simplifies considerably in the geometrically exact Cosserat micropolar case $\overline{P}\in\mathrm{SO}(3,\mathbb{R})$, in which case $p\ge1$ is already sufficient, c.f. [39].

We continue with the supercritical case which is more appropriate for macroscopic situations and closer to classical elasticity [23].

Theorem 3. (Existence for Elastic Micromorphic Model: Case 2). *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}}\in H^1(\Omega,\mathbb{R}^3)$ and $P_{\mathrm{d}}\in W^{1,1+p+q}(\Omega,\mathrm{SL}(3,\mathbb{R}))$. Moreover, let the applied external potentials satisfy (18). Then (13) with material constants conforming to case 2 admits at least one minimizing solution triple $(\varphi,\overline{P},\overline{\alpha}_p)\in H^1(\Omega,\mathbb{R}^3)\times W^{1,1+p+q}(\Omega,\mathrm{SL}(3,\mathbb{R}))\times W^{1,2+p}(\Omega,\mathbb{R})$.*

Proof. We repeat the arguments of case 1. However, the boundedness of infimizing sequences is not immediately clear. Boundedness of the microdeformations $\overline{P}^k$ holds true in the space $W^{1,1+p+q}(\Omega,\mathrm{SL}(3,\mathbb{R}))$ with $1+p+q>N=3$; hence, we may extract a subsequence, not relabeled, such that $\overline{P}^k$ converges strongly to $\widehat{\overline{P}}\in C^0(\overline{\Omega},\mathrm{SL}(3,\mathbb{R}))$ in the topology of $C^0(\overline{\Omega},\mathrm{SL}(3,\mathbb{R}))$ on account of the Sobolev-embedding theorem. Since $P^{-1,k}=e^{-\frac{\overline{\alpha}_p^k}{3}}\,\overline{P}^{-1,k}$, we obtain as well that

$$P^{-1,k}\to\widehat{P}^{-1}\in C^0(\overline{\Omega},\mathrm{GL}^+(3,\mathbb{R}))\tag{33}$$

on account of strong convergence of $\overline{\alpha}_p^k$.

Along such strongly convergent sequence of microdeformations, the sequence of deformations φ^k is also bounded in $H^1(\Omega,\mathbb{R}^3)$. However, this is not due to a basically simple estimate as in case 1, but only true after integration over the domain: at face value, we only control certain mixed symmetric expressions in the deformation gradient. Let us define $u_k\in H^{1,2}(\Omega,\mathbb{R}^3)$ by $\varphi^k=g_{\mathrm{d}}+(\varphi^k-g_{\mathrm{d}})=g_{\mathrm{d}}+u_k$. We then prove the inequality [23]

$$\begin{aligned}&\infty>I(g_{\mathrm{d}},\overline{P}_{\mathrm{d}},\overline{\alpha}_{p\mathrm{d}})>\int_\Omega W_{\mathrm{mp}}(\overline{U}_k)+W_{\mathrm{foam}}(\overline{U}_p^k,\overline{\alpha}_p^k)\\ &+W_{\mathrm{curv}}(\mathfrak{K}_{p,k},\nabla\overline{\alpha}_p^k)\,\mathrm{dV}-\Pi(\varphi_k,P^k)\\ &\ge\left(\frac{\mu_e}{8}c_K-C_2\left\|\widehat{P}^{-1}-P^{-1,k}\right\|_\infty\right)\|u_k\|^2_{H^{1,2}(\Omega)}-C\end{aligned}\tag{34}$$

where we applied the extended Korn's inequality Theorem 1 yielding the positive constant c_K for the continuous microdeformation $\widehat{P}^{-1}$.

Since $\left\|\widehat{P}^{-1} - P^{-1,k}\right\|_\infty \to 0$ for $k \to \infty$ due to (33), we are able to conclude the boundedness of u_k in $H^1(\Omega)$. Hence, φ_k is bounded in $H^1(\Omega)$. Now we obtain that $\overline{U_k} \rightharpoonup \widehat{\overline{U}} = \widehat{P}^{-1}\nabla\hat{\varphi}$ by construction with the notations as in case 1. The remainder proceeds as in case 1. This finishes the argument. The limit microdeformations $\widehat{P}$ are indeed found to be continuous.

We mention that both existence results can be easily adapted to cover the micromorphic micro-incompressible case $\overline{\alpha}_p \equiv 1$.

Conclusions

The presented variational micromorphic problem fits neatly into the framework of the direct methods of the calculus of variations. The coercivity part for the deformation is, however, nontrivial, and for the (uncommon) value of the Cosserat couple modulus $\mu_c = 0$, additional difficulties arise which can only be circumvented by the use of the generalized Korn's first inequality. In both cases 1 and 2, more realistic assumptions on the applied external loads Π are necessary to establish a lower bound for the energy I and a control of the curvature independent of the magnitude of deformation.

Altogether, the quasistatic finite micromorphic theory is established on firm mathematical grounds. With the same methods, the geometrically exact microstretch case (restricted manifold $\mathbb{R}^+ \cdot \mathrm{SO}(3,\mathbb{R})$) can also be treated.

An extension of the method to other choices of strain and curvature measures is possible. A related method has been employed for the treatment of nonlinear Cosserat shell models in [25]. Thermal stress problems for Cosserat shells have been investigated in [2]. The open case (case 3) allows for discontinuous macroscopic deformations and might therefore be a model problem allowing to describe fracture. The presented variational framework is ideally suited for subsequent numerical treatment by the finite element method.

References

1. Appell J, Zabrejko P (1990) Nonlinear superposition operators, vol 95, Cambridge tracts in mathematics. Cambridge University Press, Cambridge
2. Bîrsan M (2009) Thermal stresses in cylindrical Cosserat elastic shells. Eur J Mech A/Solid 28:94–101
3. Capriz G (1989) Continua with microstructure. Springer, Heidelberg
4. Chen Y, Lee JD (2003) Connecting molecular dynamics to micromorphic theory. I: Instantaneous and averaged mechanical variables. II: Balance laws. Physica A 322:359–376, 376–392
5. Cosserat E, Cosserat F (1909) Théorie des corps déformables. (trans: Hermann A et Fils, Paris, Theory of deformable bodies, NASA TT F-11 561, 1968)
6. Duvaut G (1970) Elasticité linéaire avec couples de contraintes: théorémes d'existence. J Mec Paris 9:325–333
7. Eringen AC (1999) Microcontinuum field theories. Springer, Heidelberg
8. Gurtin ME, Podio-Guidugli P (1992) On the formulation of mechanical balance laws for structured continua. Z Angew Math Phys 43:181–190
9. Hlavacek I, Hlavacek M (1969) On the existence and uniqueness of solutions and some variational principles in linear theories of elasticity with couple-stresses. I: Cosserat continuum. II: Mindlin's elasticity with micro-structure and the first strain gradient. J Appl Mat 14:387–426
10. Ieşan D (1971) Existence theorems in micropolar elastostatics. Int J Eng Sci 9:59–78
11. Ieşan D (2002) On the micromorphic thermoelasticity. Int J Eng Sci 40:549–568
12. Ieşan D, Pompei A (1995) On the equilibrium theory of microstretch elastic solids. Int J Eng Sci 33:399–410
13. Ieşan D, Quintanilla R (1994) Existence and continuous dependence results in the theory of microstretch elastic bodies. Int J Eng Sci 32:991–1001
14. Jeong J, Neff P (2010) Existence, uniqueness and stability in linear Cosserat elasticity for weakest curvature conditions. Math Mech Solid 15:78–95
15. Jeong J, Ramezani H, Münch I, Neff P (2009) A numerical study for linear isotropic Cosserat elasticity with conformally invariant curvature. ZAMM 89:552–569
16. Klawonn A, Neff P, Rheinbach O, Vanis S (2010) Solving geometrically exact micromorphic elasticity with a staggered algorithm. GAMM-Mitteilungen 33:57–72
17. Klawonn A, Neff P, Rheinbach O, Vanis S (2011) FETI-DP domain decomposition methods for elasticity with structural changes: P-elasticity. ESAIM: Math Mod Num Anal 45:563–602
18. Mariano PM, Modica G (2009) Ground states in complex bodies. ESAIM: Control, Optim Calc Var 15:377–402

19. Mariano PM, Stazi FL (2005) Computational aspects of the mechanics of complex materials. Arch Comput Meth Eng 12:392–478
20. Maugin GA (1998) On the structure of the theory of polar elasticity. Philos Trans R Soc Lond A 356:1367–1395
21. Neff P (2002) On Korn's first inequality with nonconstant coefficients. Proc R Soc Edinb 132A:221–243
22. Neff P (2005) On material constants for micromorphic continua. In: Wang Y, Hutter K (eds) Trends in applications of mathematics to mechanics. STAMM Proceedings (Seeheim 2004), Shaker Verlag, Aachen, pp 337–348
23. Neff P (2006) Existence of minimizers for a finite-strain micromorphic elastic solid. Proc R Soc Edinb 136A:997–1012
24. Neff P (2006) The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric. ZAMM 86:892–912
25. Neff P (2007) A geometrically exact planar Cosserat shell-model with microstructure: existence of minimizers for zero Cosserat couple modulus. Math Models Methods Appl Sci 17:363–392
26. Neff P, Chelminski K (2005) Infinitesimal elastic–plastic Cosserat micropolar theory: modelling and global existence in the rate independent case. Proc R Soc Edinb 135A:1017–1039
27. Neff P, Chelminski K (2007) Well-posedness of dynamic Cosserat plasticity. Appl Math Optim 56:19–35
28. Neff P, Fischle A, Münch I (2008) Symmetric Cauchy-stresses do not imply symmetric Biot-strains in weak formulations of isotropic hyperelasticity with rotational degrees of freedom. Acta Mech 197:19–30
29. Neff P, Forest S (2007) A geometrically exact micromorphic model for elastic metallic foams accounting for affine microstructure: modelling, existence of minimizers, identification of moduli and computational results. J Elast 87:239–276
30. Neff P, Jeong J (2009) A new paradigm: the linear isotropic Cosserat model with conformally invariant curvature energy. ZAMM 89:107–122
31. Neff P, Jeong J, Fischle A (2010) Stable identification of linear isotropic Cosserat parameters: bounded stiffness in bending and torsion implies conformal invariance of curvature. Acta Mech 211:237–249
32. Neff P, Jeong J, Münch I, Ramezani H (2009) Mean field modeling of isotropic random Cauchy elasticity versus microstretch elasticity. ZAMP 3:479–497
33. Neff P, Jeong J, Ramezani H (2009) Subgrid interaction and micro-randomness – novel invariance requirements in infinitesimal gradient elasticity. Int J Solid Struct 46:4261–4276
34. Neff P, Münch I (2008) Curl bounds grad on $SO_{(3)}$. ESAIM: Control, Optim Calc Var 14:148–159
35. Neff P, Pauly D, Witsch KJ (2011) A canonical extension of Korn's first inequality to $H(\mathrm{Curl})$ motivated by gradient plasticity with plastic spin. C R Acad Sci Paris, Ser I, doi:10.1016/j.crma.2011.10.003
36. Neff P, Pauly D, Witsch KJ (2012) Maxwell meets Korn: a new coercive inequality for tensor fields in $\mathbb{R}^{N\times N}$ with square integrable exterior derivative. Math Meth Appl Sci 35:65–71
37. Nowacki W (1986) Theory of asymmetric elasticity. Pergamon Press, Oxford
38. Pompe W (2003) Korn's first inequality with variable coefficients and its generalizations. Comment Math Univ Carolinae 44:57–70
39. Tambaca J, Velcic I (2010) Existence theorem for nonlinear micropolar elasticity. ESAIM: Control, Optim Calc Var 16:92–110
40. Tambaca J, Velcic I (2010) Semicontinuity theorem in the micropolar elasticity. ESAIM: Control, Optim Calc Var 16:337–355

Existence Theorems

► Large Existence of Solutions in Thermoelasticity Theory of Steady Vibrations

Experimental Analysis of Hot Spotting in Sliding Systems

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Overview

Systems including components with relative sliding, like clutches, brakes, hot forming tools, induce heating due to conversion of mechanical to thermal energy. The corresponding heating is a major design parameter as it influences the tribological and mechanical performances (wear of the materials, friction performances, risks of cracks, vibrations, etc.). Various types of thermal localizations may appear, usually named as hot spots, which could lead to very high local temperatures. The difficulty of understanding and modeling all of these phenomena still remains