Rank-one convexity implies polyconvexity in isotropic planar incompressible elasticity

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ABSTRACT

We study convexity properties of energy functions in plane nonlinear elasticity of incompressible materials and show that rank-one convexity of an objective and isotropic elastic energy W on the special linear group SL(2) implies the polyconvexity of W.

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RÉSUMÉ

Nous étudions les propriétés de convexité des fonctions énergie pour des matériaux incompressibles dans le cas de l'élasticité non-linéaire plane. Nous montrons que la convexité de rang 1 d'une fonction énergie isotrope et élastique W dans le groupe spécial linéaire SL(2) implique la polyconvexité de W.

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1. Introduction

The aim of this paper is to study the relation between rank-one convexity and polyconvexity of *objective* and *isotropic* real valued functions W on $\mathrm{SL}(2) = \{X \in \mathbb{R}^{2 \times 2} \mid \det X = 1\}$. These convexity properties play an important role in the theory of nonlinear hyperelasticity, where $W(\nabla \varphi)$ is interpreted as the energy density of a deformation $\varphi \colon \Omega \to \mathbb{R}^2$; here, $\Omega \subset \mathbb{R}^2$ corresponds to a planar elastic body in its reference configuration. In particular, energy functions on the domain SL(2) are used for mod-

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elling *incompressible* materials, since in this case, the deformation φ is subject to the additional constraint $\det \nabla \varphi = 1$.[1]

The notion of polyconvexity was introduced into the context of nonlinear elasticity theory by John Ball [5,6] (cf. [34,11,39]). Polyconvexity criteria in the case of spatial dimension 2 were conclusively discussed by Rosakis [36] and Šilhavý [41–44,46–48], while an exhaustive self-contained study giving necessary and sufficient conditions for polyconvexity in arbitrary spatial dimension was given by Mielke [22]. Rank-one convexity plays an important role in the existence and uniqueness theory for linear elastostatics and elastodynamics [30,16,14,15,19]. Criteria for the rank-one convexity of functions defined on $\mathrm{GL}^+(2) = \{X \in \mathbb{R}^{2\times 2} \,|\, \det X > 0\}$ were established by Knowles and Sternberg [18] as well as by Šilhavý [43,45], Dacorogna [10], Aubert [4] and Davies [12].

It is well known that the implications

$$\text{polyconvexity} \quad \Longrightarrow \quad \text{quasiconvexity} \quad \Longrightarrow \quad \text{rank-one convexity}$$

hold for functions on $\mathbb{R}^{n\times n}$ (as well as for functions on $\mathrm{SL}(n)$, see [9, Theorem 1.1]) for arbitrary dimension n. The reverse implications, on the other hand, do not hold in general: rank-one convexity does not imply polyconvexity [2] for dimension $n \geq 2$, and rank-one convexity does not imply quasiconvexity [7,49,33,11] for $n > 2$. Whether this latter implication holds for $n = 2$ is still an open question: the conjecture that rank-one convexity and quasiconvexity are *not* equivalent for $n = 2$ is also called *Morrey's conjecture* [24]. For certain classes of functions on $\mathbb{R}^{2\times 2}$, however, it has been demonstrated that the two convexity properties are, in fact, equivalent [50,40,20,49,25,8,7,32,31,27].

In a previous paper [21], we have shown that any energy function $W \colon \mathrm{GL}^+(2) \to \mathbb{R}$ which is isotropic and objective (i.e. bi-SO(2)-invariant) as well as *isochoric*[2] is rank-one convex if and only if it is polyconvex. In January 2016, a question by John Ball motivated some investigation into whether this result might be applicable to the incompressible case. In March 2016, at the Joint DMV and GAMM Annual Meeting in Braunschweig, Alexander Mielke indicated that some of his results [22] should be suitable for this task.

The main result of the present paper is Theorem 3.1, which states that for objective and isotropic energies on $\mathrm{SL}(2)$, rank-one convexity implies (and is therefore equivalent to) polyconvexity. Theorem 3.1 includes a slightly stronger two-dimensional version of a criterion by Dunn, Fosdick and Zhang (cf. Section 2.2): an energy W with $W(F) = \phi(\sqrt{\|F\| - 2})$ for $F \in \mathrm{SL}(2)$ is polyconvex on $\mathrm{SL}(2)$ (if and only if it is rank one convex) if and only if ϕ is nondecreasing and convex, regardless of any regularity assumption on the energy.[3]

2. Rank-one convexity and polyconvexity on SL(2)

We consider the concepts of *rank-one convexity* and *polyconvexity* of real-valued objective, isotropic functions W on the group $\mathrm{GL}^+(2) = \{X \in \mathbb{R}^{2\times 2} \,|\, \det X > 0\}$ and on its subgroup $\mathrm{SL}(2) = \{X \in \mathbb{R}^{2\times 2} \,|\, \det X = 1\}$. We denote by λ_1, λ_2 the singular values of F (i.e. the eigenvalues of $U = \sqrt{F^T F}$), and $\lambda_{\max} := \max\{\lambda_1, \lambda_2\}$ denotes the largest singular value of F (also called the *spectral norm* of F). The elastic energy W is assumed to be *objective* as well as *isotropic*, i.e. to satisfy the equality

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[1] Note that a function W defined only on $\mathrm{SL}(2)$ can equivalently be expressed as a (discontinuous) function $W \colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ with $W(F) = +\infty$ for all $F \notin \mathrm{SL}(2)$. This interpretation of functions not defined on all of $\mathbb{R}^{2\times 2}$ is reflected by Mielke's definition of polyconvexity [22] of energies W on $\mathrm{SL}(2)$, see Definition 2.2.

[2] A function $W \colon \mathrm{GL}^+(2) \to \mathbb{R}$ is called isochoric if $W(a\,F) = W(F)$ for all $a \in (0,\infty)$. Some relations between isotropic, objective and isochoric energies and the functions defined on $\mathrm{SL}(2)$ are discussed in Section 4. In elasticity theory, isochoric energy functions measure only the *change of form* of an elastic body, not the *change of size*.

[3] Throughout this article, $\|X\|^2 = \langle X, X \rangle$ denotes the Frobenius tensor norm of $X \in \mathbb{R}^{n\times n}$, where $\langle X, Y \rangle = \mathrm{tr}(Y^T X)$ is the standard Euclidean scalar product on $\mathbb{R}^{n\times n}$. The identity tensor on $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}(X) = \langle X, \mathbb{1} \rangle$.

$$W(Q_1\,F\,Q_2) = W(F) \quad \text{for all } F \in \mathrm{GL}^+(2) \text{ and all } Q_1, Q_2 \in \mathrm{SO}(2)\,, \tag{2.1}$$

where $\mathrm{SO}(2) = \{X \in \mathbb{R}^{2\times 2}\,|\, X^T X = \mathbb{1}\,,\; \det(X) = 1\}$ denotes the special orthogonal group.

2.1. Basic definitions

In order to discuss the different convexity conditions, we first need to define rank-one convexity as well as polyconvexity in the incompressible (planar) case, i.e. for functions on SL(2).

2.1.1. Rank-one convexity

Following a definition by Ball [5, Definition 3.2], we say that W is *rank-one convex* on $\mathrm{GL}^+(n)$ if it is convex on all closed line segments in $\mathrm{GL}^+(n)$ with end points differing by a matrix of rank one, i.e.

$$W(F + (1-\theta)\xi \otimes \eta) = W(\theta\,F + (1-\theta)\,(F + \xi \otimes \eta)) \leq \theta\,W(F) + (1-\theta)W(F + \xi \otimes \eta)$$

for all $F \in \mathrm{GL}^+(n)$, $\theta \in [0,1]$ and all $\xi, \eta \in \mathbb{R}^2$ with $F + t \cdot \xi \otimes \eta \in \mathrm{GL}^+(n)$ for all $t \in [0,1]$, where $\xi \otimes \eta$ denotes the dyadic product.

Since, in the following, we will consider the case of energy functions which are defined on only on the special linear group SL(2), we need to define rank-one convexity for functions $W\colon \mathrm{SL}(2) \to \mathbb{R}$. The restrictions imposed by rank-one convexity are less strict in this case: the functions needs to be convex only along line segments in rank-one direction which are contained in the set SL(2), i.e. satisfy the additional condition

$$\det(F + t \cdot \xi \otimes \eta) = 1 \quad \text{for all } t \in [0,1]\,. \tag{2.2}$$

The following lemma can be used to simplify condition (2.2) and thus allows us to give a simpler definition of rank-one convexity in the incompressible case.

Lemma 2.1. *Let $F, H \in \mathbb{R}^{2\times 2}$. Then*

$$\det(F + H) = \det(F) + \det(F)\,\langle F^{-T}, H \rangle + \det H\,. \tag{2.3}$$

Proof. For continuity reasons, it suffices to consider the case $\det F \neq 0$. Since, for $H \in \mathbb{R}^{2\times 2}$,

$$\det(\mathbb{1} + H) = 1 + \mathrm{tr}\,H + \det H\,, \tag{2.4}$$

we find

$$\begin{aligned}
\det(F + H) &= \det((\mathbb{1} + H\,F^{-1})\,F) = \det(\mathbb{1} + H\,F^{-1})\,\det(F)\\
&= \det(F)(1 + \mathrm{tr}(H\,F^{-1}) + \det(H\,F^{-1}))\\
&= \det(F) + \det(F)\,\langle H, F^{-T} \rangle + \det(H)\,. \quad \square
\end{aligned}$$

Since $\mathrm{rank}(\xi \otimes \eta) = 1$ implies $\det(\xi \otimes \eta) = 0$, we thus find

$$\det(F + t \cdot \xi \otimes \eta) = \det F\,[1 + t\,\langle F^{-T}, \xi \otimes \eta \rangle]$$

for $F \in \mathbb{R}^{2\times 2}$. In particular, condition (2.2) is satisfied if and only if

$$\langle F^{-T}, \xi \otimes \eta \rangle = 0\,. \tag{2.5}$$

Condition (2.5) can also be interpreted geometrically [13]: if the set $\mathrm{SL}(2)$ is regarded as a three-dimensional surface embedded in the 4-dimensional linear space $\mathbb{R}^{2\times 2}$ of all second order tensors, then the relation $D_F(\det F).H = (\det F)\langle F^{-T}, H\rangle$ implies that the tangent space $T_{\mathrm{SL}(2)}$ to $\mathrm{SL}(2)$ at $F \in \mathrm{SL}(2)$ is given by

$$T_{\mathrm{SL}(2)}(F) := \{H \in \mathbb{R}^{2\times 2} \,|\, \langle H, F^{-T}\rangle = 0\}. \tag{2.6}$$

It follows that for $F \in \mathrm{SL}(2)$,

$$\det(F + t\cdot\xi\otimes\eta) = 1 \;\; \forall t \in [0,1] \quad \Longleftrightarrow \quad \langle\xi, F^{-T}\eta\rangle = 0 \quad \Longleftrightarrow \quad \xi\otimes\eta \in T_{\mathrm{SL}(2)}(F). \tag{2.7}$$

We also note that, due to the above, (2.2) already implies that the equality $\det(F + t\cdot\xi\otimes\eta) = 1$ holds for all $t \in \mathbb{R}$ as well. These well-known (see e.g. [13]) equivalences allow for the following definition of rank-one convexity.

Definition 2.1. A function $W\colon \mathrm{SL}(2) \to \mathbb{R}$ is called *rank-one convex* if the mapping

$$t \mapsto W(F + t\cdot\xi\otimes\eta)$$

is convex on $\mathbb{R}$ for all $F \in \mathrm{SL}(2)$ and all $\xi, \eta \in \mathbb{R}^2$ such that $\xi\otimes\eta \in T_{\mathrm{SL}(2)}(F)$.

2.1.2. Polyconvexity

Throughout this article, we will use the following definitions of polyconvexity for energy functions defined on the sets on $\mathbb{R}^{n\times n}$, $\mathrm{GL}^+(n) = \{X \in \mathbb{R}^{n\times n} \,|\, \det X > 0\}$ and $\mathrm{SL}(n) = \{X \in \mathbb{R}^{n\times n} \,|\, \det X = 1\}$, respectively.

Definition 2.2.

i) (Ball [5]) A function $W\colon \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ is called polyconvex if there exists a convex function $P\colon \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ such that

$$W(F) = P(\mathbb{M}(F)) \qquad \text{for all } F \in \mathbb{R}^{n\times n}, \tag{2.8}$$

where $\mathbb{M}(F) \in \mathbb{R}^m$ denotes the vector of all minors of F.

ii) (Mielke [22]) A function $W\colon \mathrm{GL}^+(n) \to \mathbb{R}$ is called polyconvex if the function

$$\widetilde{W}\colon \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}, \quad \widetilde{W}(F) = \begin{cases} W(F) & : F \in \mathrm{GL}^+ \\ \infty & : F \notin \mathrm{GL}^+ \end{cases} \tag{2.9}$$

is polyconvex according to i).

iii) (Mielke [22]) A function $W_{\mathrm{inc}}\colon \mathrm{SL}(n) \to \mathbb{R}$ is called polyconvex if the function

$$\widetilde{W}\colon \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}, \quad \widetilde{W}(F) = \begin{cases} W_{\mathrm{inc}}(F) & : F \in \mathrm{SL}(n) \\ \infty & : F \notin \mathrm{SL}(n) \end{cases} \tag{2.10}$$

is polyconvex according to i).

2.2. Criteria for rank-one convexity and polyconvexity in the incompressible planar case

For twice differentiable energies on SL(3), necessary and sufficient conditions for rank-one convexity were established by Zubov and Rudev [53,52] as well as by Zee and Sternberg [51]. An easily applicable criterion for rank-one convexity is available for the special case of differentiable functions on SL(3) of the form $F \mapsto W(F) = \phi(\gamma)$, where $\gamma = \sqrt{\|F\|^2 - 3}$ represents the *amount of shear* (cf. Theorem 3.1): Dunn, Fosdick and Zhang [13] have shown that the energy W is rank-one convex on SL(3) if and only if ϕ is nondecreasing and convex. This criterion is related, with appropriate modifications, to those obtained by Zee and Sternberg [51, p. 83], but it only requires the energy to be once differentiable. Note that not every function on SL(3) can be written in the form $W(F) = \phi(\sqrt{\|F\|^2 - 3})$, so this criterion cannot be applied in the general case of incompressible energies (as was already noted in [13]). In contrast to the three-dimensional case, every energy defined on SL(2) admits a unique representation in terms of the amount of shear $\sqrt{\|F\|^2 - 2}$. This representation was also used by Mielke [22] in order to establish necessary and sufficient conditions for polyconvexity on SL(2), cf. Proposition 2.3.

The following necessary and sufficient conditions for rank-one convexity on SL(2) are adapted from a similar criterion for the ellipticity[4] of incompressible, isotropic hyperelastic solids by Abeyaratne [1]. A proof of the Proposition is given in Appendix A.

Proposition 2.1. *Let* $W \colon \mathrm{SL}(2) \to \mathbb{R}$ *be a twice-differentiable objective and isotropic function. Then there exists a unique function* $\psi \colon [0, \infty) \to \mathbb{R}$ *such that*

$$W(F) = \psi(I), \qquad I = \|F\|^2 = \lambda_{\max}^2(F) + \frac{1}{\lambda_{\max}^2(F)} \tag{2.11}$$

for all $F \in \mathrm{SL}(2)$, *where* $\lambda_{\max}(F)$ *is the largest singular value of* F. *Furthermore, the following are equivalent:*

i) *W is rank-one convex,*
ii) *ψ satisfies the inequalities*

$$\frac{d\psi}{dI}(I) \geq 0, \quad 2\,(I-2)\,\frac{d^2\psi}{dI^2}(I) + \frac{d\psi}{dI}(I) \geq 0 \quad \text{for all } I \in [2, \infty). \tag{2.12}$$

In the two-dimensional incompressible case, i.e. for an objective and isotropic energy W on SL(2), another representation of the energy can be obtained from formula (2.11): since, for $F \in \mathrm{SL}(2)$,

$$\gamma := \sqrt{\|F\|^2 - 2} = \sqrt{\lambda_1^2 + \lambda_2^2 - 2} = \sqrt{(\lambda_1 - \lambda_2)^2} = |\lambda_1 - \lambda_2| = \lambda_{\max} - \lambda_{\min} = \lambda_{\max} - \frac{1}{\lambda_{\max}}$$

and

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[4] Abeyaratne [1] considers the *ordinary ellipticity* of twice-differentiable energies, which are defined as follows:

for compressible materials: $\det Q \neq 0$ (or $\langle Q\,\xi, \xi \rangle \neq 0$ for all $\xi \in \mathbb{R}^2 \setminus \{0\}$),

for incompressible materials: $\det \begin{pmatrix} Q_{11} & Q_{12} & -m_1 \\ Q_{21} & Q_{22} & -m_2 \\ m_1 & m_2 & 0 \end{pmatrix} \neq 0$ (or $\langle Q\,\xi, \xi \rangle \neq 0$ for all $\xi \in \mathbb{R}^2 \setminus \{0\}$ with $\langle \xi, F^{-T}\eta \rangle = 0$),

where $Q_{\alpha\gamma} = \sum_{\beta,\delta=1,2} \frac{\partial^2 W}{\partial F_{\alpha\beta}\,\partial F_{\gamma\delta}}\,\eta_\beta \eta_\delta$ with $\alpha, \gamma \in \{1, 2\}$ is the *acoustic tensor*, $m = F^{-T}\eta$ and $\eta \in \mathbb{R}^2 \setminus \{0\}$. Abeyaratne's mechanical motivation is the requirement that the system of the jump equations of equilibrium should be satisfied only by the trivial solution. In the three-dimensional case, this concept was also considered by Zee and Sternberg [51]. We recall that for compressible materials, the *strong ellipticity* (or *strict Legendre–Hadamard ellipticity*) is equivalent to the positive definiteness of the acoustic tensor Q, while rank-one convexity is equivalent to the positive semidefiniteness of Q.

$$I = 2 + \left(\lambda_{\max} - \frac{1}{\lambda_{\max}} \right)^2 = 2 + \gamma^2 \,,$$

there exists a unique function $\phi \colon [0, \infty) \to \mathbb{R}$ such that

$$W(F) = \psi(I) = \phi(\sqrt{\|F\|^2 - 2}) = \phi\left(\lambda_{\max} - \frac{1}{\lambda_{\max}} \right) \quad \text{for all } F \in \mathrm{SL}(2) \,. \tag{2.13}$$

The next criterion for rank-one convexity in terms of this representation can be obtained by a direct adaptation of the proof of the aforementioned three-dimensional result by Dunn, Fosdick and Zhang [13] to the two-dimensional case.

Proposition 2.2. *Let* $W \colon \mathrm{SL}(2) \to \mathbb{R}$ *be an objective and isotropic differentiable function. Then there exists a unique function* $\phi \colon [0, \infty) \to \mathbb{R}$ *such that*

$$W(F) = \phi\left(\lambda_{\max}(F) - \frac{1}{\lambda_{\max}(F)} \right)$$

for all $F \in \mathrm{SL}(2)$, *where* $\lambda_{\max}(F)$ *is the largest singular value of* F. *Furthermore, the following are equivalent:*

 i) W *is rank-one convex,*
 ii) ϕ *is nondecreasing and convex on* $[0, \infty)$.

It is easy to see that if an energy (and thus ψ) is twice differentiable, then Proposition 2.2 and Proposition 2.1 are equivalent: for $\gamma = \sqrt{I - 2}$ we find

$$\frac{d\psi}{dI} = \frac{d\phi}{d\gamma} \frac{1}{2\sqrt{I-2}} = \frac{1}{\gamma} \frac{d\phi}{d\gamma} \quad \text{and} \quad \frac{d^2\psi}{dI^2} = \frac{1}{4\gamma^2} \frac{d^2\phi}{d\gamma^2} - \frac{1}{4\gamma^3} \frac{d\phi}{d\gamma} \,, \tag{2.14}$$

thus the monotonicity of ϕ in Proposition 2.1 is equivalent to $\frac{d\psi}{dI}(I) \geq 0$ for all $I \in [2, \infty)$, while the convexity of ϕ is equivalent to $2(I-2)\frac{d^2\psi}{dI^2}(I) + \frac{d\psi}{dI}(I) \geq 0$ for all $I \in [2, \infty)$.

In addition to these criteria for rank-one convexity, we will use the following polyconvexity criterion, which is due to Mielke [22, Theorem 5.1].

Proposition 2.3. *Let* $W \colon \mathrm{SL}(2) \to \mathbb{R}$ *be an objective and isotropic function, and* $\phi \colon [0, \infty) \to \mathbb{R}$ *the unique function with*

$$W(F) = \phi\left(\lambda_{\max}(F) - \frac{1}{\lambda_{\max}(F)} \right)$$

for all $F \in \mathrm{SL}(2)$, *where* $\lambda_{\max}(F)$ *is the largest singular value of* F. *The following are equivalent:*

 i) ϕ *is nondecreasing and convex on* $[0, \infty)$,
 ii) W *is polyconvex (in the sense of Definition 2.2 iii)).*

Using Mielke's criterion as well as representation formula (2.11), Mora-Corral and Strugaru [23] recently gave a criterion for polyconvexity on $\mathrm{SL}(2)$ with respect to the representation of the energy in terms of the largest singular value.

3. Equivalence of rank-one convexity and polyconvexity on SL(2)

We now want to show that for objective and isotropic energy functions on SL(2), rank-one convexity and polyconvexity are equivalent.

3.1. Differentiable functions

For *differentiable* functions, this result can be obtained directly by comparing Proposition 2.2 and Proposition 2.3.

Proposition 3.1. *Let* $W\colon \mathrm{SL}(2) \to \mathbb{R}$ *be objective and isotropic as well as* differentiable. *Then the following are equivalent:*

i) W *is rank-one convex,*

ii) *the function* $\phi\colon [0,\infty) \to \mathbb{R}$ *with* $W(F) = \phi\left(\lambda_{\max}(F) - \dfrac{1}{\lambda_{\max}(F)}\right)$ *is nondecreasing and convex,*

iii) W *is polyconvex.*

3.2. The general case

Our main result of this paper is the equivalence of rank-one convexity and polyconvexity for objective and isotropic energy functions in general, without any regularity assumptions. The following theorem also provides another geometric interpretation of the criteria from Proposition 2.2 and Proposition 2.3 by Dunn et al. and Mielke: the convexity and monotonicity of the function ϕ is equivalent to the convexity of the energy with respect to the *amount of shear*. Note that in the planar incompressible case, an energy function is already completely determined by its response to simple shear deformations [1].[5]

Theorem 3.1. *Let* $W\colon \mathrm{SL}(2) \to \mathbb{R}$ *be an objective and isotropic function. Then the following are equivalent:*

i) W *is rank-one convex,*

ii) W *is polyconvex,*

iii) *the mapping* $\widetilde{\phi}\colon \mathbb{R} \to \mathbb{R}$, $\widetilde{\phi}(\gamma) = W\left(\left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right)\right)$ *is convex,*

iv) *the function* $\phi\colon [0,\infty) \to \mathbb{R}$ *with* $W(F) = \phi(\sqrt{\|F\|^2 - 2}) = \phi\left(\lambda_{\max}(F) - \dfrac{1}{\lambda_{\max}(F)}\right)$ *is nondecreasing and convex.*

Proof. i) $\implies$ iii):

We note that

$$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} = \mathbb{1} + \gamma \cdot \xi \otimes \eta \qquad \text{with} \quad \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Furthermore, $\det(\mathbb{1} + \gamma \cdot \xi \otimes \eta) = 1$ for all $\gamma \in \mathbb{R}$. Thus the rank-one convexity of W implies that the mapping $\gamma \mapsto W\left(\left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right)\right) = W(\mathbb{1} + \gamma \cdot \xi \otimes \eta)$ is convex on $\mathbb{R}$.

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[5] Any plane volume preserving deformation can be decomposed locally into the product of a simple shear in a suitable direction followed or preceded by a suitable rotation. More precisely, for any $F \in \mathrm{SL}(2)$, there exist $Q_1, Q_2 \in \mathrm{O}(2)$ such that

$$F = Q_1\, K\, Q_2, \qquad K = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \qquad \gamma = \pm\sqrt{I - 2}, \qquad I = \|F\|^2.$$

iii) $\implies$ iv):

Let $\phi\colon [0,\infty) \to \mathbb{R}$ denote the uniquely defined function with $W(F) = \phi(\lambda_{\max}(F) - \frac{1}{\lambda_{\max}(F)})$ for all $F \in \mathrm{SL}(2)$. We first show that $\phi(t) = \widetilde{\phi}(t)$ for all $t \geq 0$: for $\gamma \geq 0$, the singular values of the simple shear are

$$\lambda_{\max}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}) = \frac{1}{2}(\gamma + \sqrt{\gamma^2 + 4}) \qquad \text{and} \qquad \lambda_{\min}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}) = \frac{1}{2}(-\gamma + \sqrt{\gamma^2 + 4}). \tag{3.1}$$

Thus we find

$$\lambda_{\max}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}) - \left(\lambda_{\max}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix})\right)^{-1} = \lambda_{\max}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}) - \lambda_{\min}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix})$$

$$= \frac{1}{2}(\gamma + \sqrt{\gamma^2 + 4}) - \frac{1}{2}(-\gamma + \sqrt{\gamma^2 + 4}) = \gamma \tag{3.2}$$

and therefore

$$\widetilde{\phi}(\gamma) = W(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}) = \phi\left(\lambda_{\max}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}) - \left(\lambda_{\max}(\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix})\right)^{-1}\right) = \phi(\gamma)$$

for all $\gamma \geq 0$.

Since $\widetilde{\phi}$ is convex by assumption of condition iii), it follows that $\phi = \widetilde{\phi}|_{[0,\infty)}$ is convex on $[0,\infty)$ as well. Thus it only remains to show that $h = \widetilde{\phi}|_{[0,\infty)}$ is also nondecreasing.

Let $0 \leq t_1 \leq t_2$. Then t_1 lies in the convex hull of $-t_2$ and t_2, i.e. $t_1 = s(-t_2) + (1-s)t_2$ for some $s \in [0,1]$. Since $\widetilde{\phi}$ is convex on $\mathbb{R}$ and $\widetilde{\phi}(-t) = \widetilde{\phi}(t)$ for all $t \in \mathbb{R}$, we thus find

$$\phi(t_1) = \widetilde{\phi}(t_1) \leq s\,\widetilde{\phi}(-t_2) + (1-s)\,\widetilde{\phi}(t_2) = \widetilde{\phi}(t_2) = \phi(t_2),$$

which shows that ϕ is nondecreasing.

iv) $\implies$ ii):

Condition iv), i.e. the convexity and monotonicity of ϕ, is exactly condition i) in Proposition 2.3, which immediately implies that W is polyconvex. Note that in contrast to Proposition 2.2, Proposition 2.3 does not require the energy to be differentiable.

ii) $\implies$ i):

This implication is well known, see for example [11, Theorem 5.3]. $\quad\square$

Remark 3.1. In addition to showing the equivalence between rank-one convexity and polyconvexity on $\mathrm{SL}(2)$, Theorem 3.1 requires no regularity of the energy and thus improves the known criteria for rank-one convexity on $\mathrm{SL}(2)$.

Remark 3.2. Note that we have only used the implication i) $\Rightarrow$ ii) from Proposition 2.3. Since the composition of a polyconvex function with a monotone and convex function is polyconvex, in order to prove this implication it is sufficient to show that the function

$$W\colon \mathrm{SL}(2) \to \mathbb{R}, \quad W(F) = \lambda_{\max} - \frac{1}{\lambda_{\max}}$$

is polyconvex on $\mathrm{SL}(2)$. We will give an alternative proof of this statement in Appendix B.
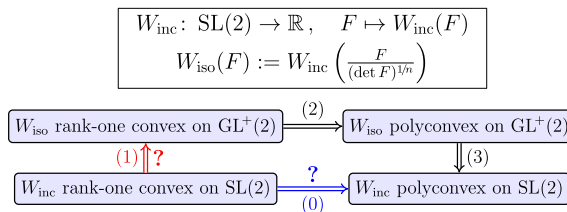
Fig. 1. A visualization of our first approach: implications (2) and (3) hold (see Proposition 4.2 and Proposition 4.1), whereas it turns out from Remark 4.1 that implication (1) does not hold in general. Implication (0) is the main result of this article (Theorem 3.1).

4. Functions on SL(2) and isochoric functions on $\mathrm{GL}^+(2)$

Functions on the special linear group $\mathrm{SL}(2)$ are closely connected to so-called *isochoric* functions on $\mathrm{GL}^+(2)$, i.e. functions $W_{\mathrm{iso}}\colon \mathrm{GL}^+(2) \to \mathbb{R}$ with $W_{\mathrm{iso}}(a\,F) = W(F)$ for all $a \in \mathbb{R}^+$. In particular, any isochoric function can be written as [21]

$$W_{\mathrm{iso}}(F) = W_{\mathrm{inc}}\left(\frac{F}{(\det F)^{1/2}}\right), \tag{4.1}$$

where $W_{\mathrm{inc}} = W\big|_{\mathrm{SL}(2)}$ is the restriction of W_{iso} to $\mathrm{SL}(2)$. Furthermore, the relation (4.1) describes a *bijection* between the set of isochoric functions and the set of functions on $\mathrm{SL}(2)$. We also note that W_{inc} is objective/isotropic if and only if W_{iso} is objective/isotropic.

A result similar to Theorem 3.1 has previously been shown to hold for isochoric functions [21]. In the following, we briefly discuss a failed first attempt to prove Theorem 3.1 by using this earlier result, thereby highlighting the difference between convexity properties of isochoric functions and functions on $\mathrm{SL}(2)$.

Proposition 4.1 *([21]). Let $W_{iso}\colon \mathrm{GL}^+(2) \to \mathbb{R}$ be an objective, isotropic and isochoric function, i.e.*

$$W_{iso}(a\,F) = W_{iso}(F) \quad \text{for all } a \in \mathbb{R}^+ := (0,\infty),$$

and let $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $h\colon \mathbb{R}^+ \to \mathbb{R}$ denote the uniquely determined functions with

$$W_{iso}(F) = g(\lambda_1, \lambda_2) = h\left(\frac{\lambda_1}{\lambda_2}\right) = h\left(\frac{\lambda_2}{\lambda_1}\right) \tag{4.2}$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. Then the following are equivalent:

i) *W_{iso} is polyconvex.*
ii) *W_{iso} is rank-one convex,*
iii) *g is separately convex,*
iv) *h is convex on $\mathbb{R}^+$,*
v) *h is convex and non-decreasing on $[1, \infty)$.*

Of course, in order to show the equivalence of rank-one convexity and polyconvexity for functions on $\mathrm{SL}(2)$, one might attempt to combine Proposition 4.1 with the relation (4.1).

This approach, which is visualized in Fig. 1, can be summarized as follows: we would like to show that the rank-one convexity of W_{inc} implies the rank-one convexity of W_{iso} (in the notation of (4.1)). If this was the case, then we could apply Theorem 4.1 to show that W_{iso} is polyconvex, and thus W_{inc} is polyconvex as the restriction of the polyconvex function W_{iso} to $\mathrm{SL}(2)$, cf. Definition 2.2.

However, this approach turned out not to be viable: although the rank-one convexity of W_{iso} implies the rank-one convexity of W_{inc}, the reverse is not true in general.

Proposition 4.2. *Let $W_{iso}\colon \mathrm{GL}^+(2) \to \mathbb{R}$ be an objective, isotropic and isochoric function. Then rank-one convexity (equivalently polyconvexity) of W_{iso} on $\mathrm{GL}^+(2)$ implies rank-one convexity (equivalently polyconvexity) of $W_{inc}\colon \mathrm{SL}(2) \to \mathbb{R}$ on $\mathrm{SL}(2)$. The reverse implication does not hold in general.*

Proof. Since $W_{\mathrm{iso}}\colon \mathrm{GL}^+(2) \to \mathbb{R}$ is an objective, isotropic, isochoric rank-one convex function on $\mathrm{GL}^+(2)$, the unique function $h\colon \mathbb{R}^+ \to \mathbb{R}$ satisfying (4.2) is convex and non-decreasing on $[1, \infty)$. For all $F \in \mathrm{SL}(2)$,

$$h\left(\lambda_{\max}^2(F)\right) = h\left(\frac{\lambda_{\max}(F)}{\lambda_{\min}(F)}\right) = W_{\mathrm{iso}}(F) = W_{\mathrm{inc}}(F) = \phi\left(\lambda_{\max}(F) - \frac{1}{\lambda_{\max}(F)}\right), \qquad (4.3)$$

where $\lambda_{\max}(F)$ is the largest singular value of F and $\phi\colon [0, \infty) \to \mathbb{R}$ is the unique function such that the last equality of (4.3) holds. Therefore

$$\phi(\theta) = h\left(\frac{(\theta + \sqrt{\theta + 4})^2}{4}\right) \qquad \text{for all } \theta \geq 0. \qquad (4.4)$$

Since the mapping $\theta \mapsto \frac{(\theta + \sqrt{\theta + 4})^2}{4}$ is monotone increasing and convex from $[0, \infty)$ to $[1, \infty)$ and $h\colon \mathbb{R}^+ \to \mathbb{R}$ is convex and non-decreasing on $[1, \infty)$, the function $\phi\colon [0, \infty) \to \mathbb{R}$ is nondecreasing and convex. Thus Theorem 3.1 yields the polyconvexity and rank-one convexity of the function $W_{\mathrm{inc}} = W|_{\mathrm{SL}(2)}$. For the second part of the proof, we refer to Remark 4.1 for a counterexample. $\square$

Remark 4.1 *(Counterexample to the above approach).* Consider the function $W_{\mathrm{iso}}\colon \mathrm{GL}^+(2) \to \mathbb{R}$ with

$$W_{\mathrm{iso}}(F) = \left|\sqrt{\frac{\lambda_1}{\lambda_2}} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right|$$

for all $F \in \mathrm{GL}^+(2)$ with singular values $\lambda_1, \lambda_2 \in \mathbb{R}^+$. Then

i) W_{iso} is objective, isotropic and isochoric on $\mathrm{GL}^+(2)$,
ii) W_{iso} is *not* rank-one convex on $\mathrm{GL}^+(2)$,
iii) the restriction $W_{\mathrm{inc}} = W_{\mathrm{iso}}|_{\mathrm{SL}(2)}$ of W_{iso} to $\mathrm{SL}(2)$ is polyconvex and rank-one convex on $\mathrm{SL}(2)$.

Proof. In order to show i), it suffices to remark that

$$W_{\mathrm{iso}}(F) = h\left(\frac{\lambda_1}{\lambda_2}\right) \qquad \text{with} \qquad h(t) = \left|\sqrt{t} - \sqrt{\frac{1}{t}}\right|$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. Thus W_{iso} is objective, isotropic and isochoric, and according to Theorem 4.1, W_{iso} is rank-one convex if and only if h is convex and non-decreasing on $[1, \infty)$. Since

$$h''(t) = -\frac{1}{4} t^{-\frac{3}{2}} - \frac{3}{4} t^{-\frac{5}{2}}$$

for all $t > 1$, h is not convex, which proves ii).

It remains to show iii), i.e. that the restriction $W_{\mathrm{inc}} = W_{\mathrm{iso}}|_{\mathrm{SL}(2)}$ of W_{iso} to $\mathrm{SL}(2)$ is (SL-)polyconvex. We first give an explicit representation of W_{inc}: let $F \in \mathrm{SL}(2)$. Then $\frac{1}{\lambda_2} = \lambda_1 =: \lambda$, thus

$$W_{\mathrm{inc}}(F) = W_{\mathrm{iso}}(F) = \left|\sqrt{\frac{\lambda_1}{\lambda_2}} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right| = \left|\sqrt{\frac{\lambda}{\frac{1}{\lambda}}} - \sqrt{\frac{\frac{1}{\lambda}}{\lambda}}\right| = \left|\sqrt{\lambda^2} - \sqrt{\frac{1}{\lambda^2}}\right| = \left|\lambda - \frac{1}{\lambda}\right|$$

$$= \max\{\lambda, \tfrac{1}{\lambda}\} - \min\{\lambda, \tfrac{1}{\lambda}\} = \max\{\lambda_1, \lambda_2\} - \min\{\lambda_1, \lambda_2\}$$

$$= \lambda_{\max} - \frac{1}{\lambda_{\max}} = \phi\left(\lambda_{\max} - \frac{1}{\lambda_{\max}}\right),$$

where $\lambda_{\max} = \max\{\lambda_1, \lambda_2\} = \max\{\lambda, \tfrac{1}{\lambda}\}$ and $\phi\colon [0,\infty) \to \mathbb{R}$ is defined by $\phi(s) = s$. According to Proposition 2.3, a function $W\colon \mathrm{SL}(2) \to \mathbb{R}$ with $W(F) = \phi(\lambda_{\max} - \frac{1}{\lambda_{\max}})$ for all $F \in \mathrm{SL}(2)$ with maximum singular value $\lambda_{\max}$ is polyconvex if and only if ϕ is non-decreasing and convex on $[0,\infty)$. Since the mapping ϕ obviously satisfies these conditions, the function W_{inc} is polyconvex on $\mathrm{SL}(2)$ and thus rank-one convex [11, Theorem 5.3] (cf. Theorem 3.1). $\square$

5. Outlook

We finish with some open questions: consider an energy $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ with volumetric-isochoric split[6]

$$W(F) = W_{\mathrm{iso}}\left(\frac{F}{\det F^{1/2}}\right) + W_{\mathrm{vol}}(\det F). \tag{5.1}$$

Such a split is relevant for slightly compressible materials like vulcanized rubber, cf. [3,17,28,29,26]. It is an open question whether rank-one convexity implies polyconvexity for an energy W of this type. There might also be additional restrictions on the isochoric part $W_{\mathrm{iso}}(\frac{F}{\det F^{1/2}})$ and the volumetric part $W_{\mathrm{vol}}(\det F)$ which assure that the implication holds.

A related question is whether the rank-one convexity of the total energy W implies the rank-one convexity of the individual parts $W_{\mathrm{iso}}(\frac{F}{\det F^{1/2}})$ and $W_{\mathrm{vol}}(\det F)$, respectively. In particular, this would imply (due to Theorem 3.1 and Proposition 4.1) that rank-one convexity is equivalent to polyconvexity for energy functions of the form (5.1).

Acknowledgements

Appendix A. An alternative proof of a rank-one convexity criterion for twice-differentiable functions on SL(2)

In this appendix, we will give an alternative proof of the rank-one convexity criterion stated in Proposition 2.1. We assume that the energy $W\colon \mathrm{SL}(2) \to \mathbb{R}$ is twice-differentiable. In this case, rank-one convexity is equivalent to *Legendre–Hadamard ellipticity* on $\mathrm{SL}(2)$:

$$D_F^2 W(F)(\xi \otimes \eta, \xi \otimes \eta) \geq 0 \quad \text{for all } F \in \mathrm{SL}(2) \text{ and } \xi, \eta \in \mathbb{R}^2 \text{ with } \xi \otimes \eta \in T_{\mathrm{SL}(2)}(F). \tag{A.1}$$

---

[6] The applications of a split of the form (5.1) in the anisotropic case has been discussed by Sansour [38]. Sansour's statement for isotropy is already contained in a 1948 article by Richter [35, page 209].

The Legendre–Hadamard ellipticity condition can equivalently be stated as

$$\langle Q(F, \eta)\, \xi, \, \xi \rangle \geq 0 \quad \text{for all} \ \ F \in \mathrm{SL}(2) \ \ \text{and} \ \ \xi, \eta \in \mathbb{R}^2 \ \ \text{such that} \ \ \langle F^{-T}\eta, \, \xi \rangle = 0\,, \tag{A.2}$$

where the acoustic tensor $Q = (Q_{\alpha\gamma})_{\alpha\gamma}$ is defined by

$$Q_{\alpha\gamma}(F, \eta) = \frac{\partial^2 W(F)}{\partial F_{\alpha\beta}\, \partial F_{\gamma\delta}}\, \eta_\beta \eta_\delta\,. \tag{A.3}$$

Here, we employ the Einstein summation convention for Greek subscripts (which take the values $1, 2$).

Note that for $\eta = 0$, the Legendre–Hadamard ellipticity condition is satisfied for all $F \in \mathrm{SL}(2)$ and all $\xi \in \mathbb{R}^2$. Hence we may assume that η is a unit vector. Note also that for all $\eta \in \mathbb{R}^2 \setminus \{0\}$ with $\|\eta\| = 1$ and all $F \in \mathrm{SL}(2)$,

$$\langle F^{-T}\eta, \, \xi \rangle = 0 \quad \Longleftrightarrow \quad \xi = a\, \epsilon\, F^{-T}\eta \quad \text{for some} \ \ a \in \mathbb{R}\,, \tag{A.4}$$

where

$$\epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.5}$$

is the two-dimensional alternator. Since $F^{-1} = \epsilon^T F \epsilon$ for $F \in \mathrm{SL}(2)$ and $\epsilon\, \epsilon^T = \mathbb{1}$, we find that condition (A.2) is equivalent to

$$\langle Q(F, \eta)\, F^T \epsilon\, \eta, \, F^T \epsilon\, \eta \rangle \geq 0 \quad \text{for all} \ \ F \in \mathrm{SL}(2) \ \ \text{and} \ \ \eta \in \mathbb{R}^2 \setminus \{0\} \ \ \text{with} \ \ \|\eta\| = 1\,, \tag{A.6}$$

which can be written in terms of the components as

$$\epsilon_{\alpha\lambda} \epsilon_{\beta\mu} F_{\gamma\lambda} F_{\delta\mu} Q_{\gamma\delta} \eta_\alpha \eta_\beta \geq 0 \quad \text{for all} \ \ F \in \mathrm{SL}(2) \ \ \text{and} \ \ \eta \in \mathbb{R}^2 \setminus \{0\} \ \ \text{with} \ \ \|\eta\| = 1\,. \tag{A.7}$$

Now consider the fourth order elasticity tensor defined by

$$\mathbb{C}_{\alpha\beta\gamma\delta} = \frac{\partial^2 W(F)}{\partial F_{\alpha\beta} \partial F_{\gamma\delta}}\,. \tag{A.8}$$

The Legendre–Hadamard condition on $\mathrm{SL}(2)$ is equivalent to

$$\langle \mathbb{C}.\, (F^T \epsilon\, \eta) \otimes \eta, \, (F^T \epsilon\, \eta) \otimes \eta \rangle \geq 0 \quad \text{for all} \ \ F \in \mathrm{SL}(2) \ \ \text{and} \ \ \eta \in \mathbb{R}^2 \setminus \{0\} \ \ \text{with} \ \ \|\eta\| = 1\,. \tag{A.9}$$

For objective and isotropic energies $W \colon \mathrm{SL}(2) \to \mathbb{R}$ given by

$$W(F) = \psi(I)\,, \qquad I = \|F\|^2 = \lambda_{\max}^2(F) + \frac{1}{\lambda_{\max}^2(F)}\,, \tag{A.10}$$

we find

$$\frac{\partial W(F)}{\partial F_{\alpha\beta}} = 2\, F_{\alpha\beta}\, \psi'(I)\,, \qquad \mathbb{C}_{\alpha\beta\gamma\delta} = 4\, F_{\alpha\beta}\, F_{\gamma\delta}\, \psi''(I) + 2\, \delta_{\alpha\gamma}\, \delta_{\beta\delta}\, \psi'(I)\,, \tag{A.11}$$

and the acoustic tensor is given by

$$Q_{\alpha\gamma} = \mathbb{C}_{\alpha\beta\gamma\delta}\,\eta_\beta\eta_\delta = 4\,F_{\alpha\beta}\,F_{\gamma\delta}\,\eta_\beta\,\eta_\delta\,\psi''(I) + 2\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,\eta_\beta\,\eta_\delta\,\psi'(I)$$
$$= 4\,F_{\alpha\beta}\,\eta_\beta\,F_{\gamma\delta}\eta_\delta\,\psi''(I) + 2\,\delta_{\alpha\gamma}\,\eta_\delta\,\eta_\delta\,\psi'(I). \tag{A.12}$$

Hence, after some calculations (cf. [1]), condition (A.6) becomes

$$(\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}C_{\alpha\beta}\,\eta_\lambda\eta_\mu)\,\psi'(I) + 2(\epsilon_{\alpha\lambda}C_{\alpha\rho}\,\eta_\rho\eta_\lambda)^2\psi''(I) \geq 0 \quad \text{for all }\ F \in \mathrm{SL}(2)\,,\ \ \eta \in \mathbb{R}^2\setminus\{0\}\,,\ \|\eta\| = 1\,,$$

where $C = F^T F$. Due to isotropy, we can assume without loss of generality that

$$C = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}, \tag{A.13}$$

thus we obtain the condition

$$(\lambda_1^2\eta_2^2 + \lambda_2^2\,\eta_1^2)^2\ \psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\ \eta_1^2\eta_2^2\,\psi''(I) \geq 0 \tag{A.14}$$
$$\text{for all }\ F \in \mathrm{SL}(2)\,,\ \ \eta \in \mathbb{R}^2\setminus\{0\}\,,\ \ \|\eta\| = 1\,,$$

which in turn is equivalent to

$$[\lambda_2^2\,\psi'(I)]\,\eta_1^4 + [\lambda_1^2\,\psi'(I)]\,\eta_2^4 + [(\lambda_1^2 + \lambda_2^2)\psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\,\psi''(I)]\,\eta_1^2\,\eta_2^2\ \geq\ 0 \tag{A.15}$$
$$\text{for all }\ F \in \mathrm{SL}(2)\,,\ \ \eta \in \mathbb{R}^2\setminus\{0\}\,,\ \ \|\eta\| = 1\,.$$

Using the notation [1]

$$E_{11} = \lambda_2^2\,\psi'(I)\,, \quad E_{22} = \lambda_1^2\,\psi'(I)\,, \quad E_{12} = E_{21} = \frac{1}{2}[(\lambda_1^2 + \lambda_2^2)\psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\,\psi''(I)]\,,$$

condition (A.14) can be written as

$$\langle E\,\zeta,\,\zeta\rangle \geq 0 \quad \text{for all }\ \zeta \in \mathbb{R}_+^2\,. \tag{A.16}$$

Note carefully that (A.16) does not imply that E is positive semi-definite, since the inequality needs to hold only for $\zeta \in \mathbb{R}_+^2$ and not for all $\zeta \in \mathbb{R}^2$. Instead, it is easy to see that condition (A.16) holds if and only if

$$E_{11} \geq 0 \quad \text{and} \quad E_{22} \geq 0 \quad \text{and} \quad \big[E_{12} < 0 \implies \det E \geq 0\big]\,, \tag{A.17}$$

which, after some computation, can be stated as

$$\psi'(I) \geq 0 \quad \text{and} \quad \Big[(\lambda_1^2 + \lambda_2^2)\,\psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\,\psi''(I) < 0 \tag{A.18}$$
$$\implies \quad (\lambda_1 + \lambda_2)^2\,\Psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\,\Psi''(I) \geq 0\Big]\,.$$

Since under the assumption $\psi'(I) \geq 0$, the implication

$$(\lambda_1^2 + \lambda_2^2)\,\psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\,\psi''(I) \geq 0 \tag{A.19}$$
$$\implies \quad (\lambda_1 + \lambda_2)^2\,\Psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\,\Psi''(I) \geq 0$$

holds in general, we conclude that condition (A.19) (and thus the Legendre–Hadamard ellipticity of W) is equivalent to

$$\psi'(I) \geq 0 \quad \text{and} \quad (\lambda_1 + \lambda_2)^2\,\Psi'(I) + 2(\lambda_1^2 - \lambda_2^2)^2\,\Psi''(I) \geq 0\,,$$

which we can write as

$$\psi'(I) \geq 0\,, \qquad \psi'(I) + 2(\lambda_1 - \lambda_2)^2\,\psi''(I) \geq 0\,.$$

Finally, in terms of I, this can be expressed as

$$\psi'(I) \geq 0\,, \qquad \psi'(I) + 2(I - 2)\,\psi''(I) \geq 0\,,$$

which is the criterion for rank-one convexity given in Proposition 2.2 and Proposition 2.1.

Appendix B. Polyconvexity of the mapping $F \mapsto \lambda_{\max} - \frac{1}{\lambda_{\max}}$

In order to keep the proof of our main result self-contained (see Remark 3.2), we will show that the mapping $F \mapsto \lambda_{\max} - \frac{1}{\lambda_{\max}}$ is polyconvex on $\mathrm{SL}(2)$. First, we require the following lemma, which states that every polyconvex function on $\mathrm{SL}(2)$ is the restriction of a convex function on $\mathbb{R}^{2\times2}$ to $\mathrm{SL}(2)$.

Lemma B.1. *A function $W\colon \mathrm{SL}(2) \to \mathbb{R}$ is polyconvex if and only if there exists a convex function $\widetilde{P}\colon \mathbb{R}^{2\times2} \to \mathbb{R} \cup \{\infty\}$ such that $W(F) = \widetilde{P}(F)$ for all $F \in \mathrm{SL}(2)$.*

Proof. First, assume that $W(F) = \widetilde{P}(F)$ for all $F \in \mathrm{SL}(2)$ with a convex function $\widetilde{P}\colon \mathbb{R}^{2\times2} \to \mathbb{R} \cup \{\infty\}$. Then the function

$$P\colon \mathbb{R}^{2\times2} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}\,, \quad P(M,\delta) := \begin{cases} \widetilde{P}(M) & : \ \delta = 1\,, \\ \infty & : \ \delta \neq 1 \end{cases}$$

is convex and

$$W(F) = \widetilde{P}(F) = P(F,1) = P(F, \det F) \qquad \text{for all } \ F \in \mathrm{SL}(2)$$

as well as

$$W(F) = \infty = P(F, \det F) \qquad \text{for all } \ F \notin \mathrm{SL}(2)\,.$$

Thus, according to Definition 2.2, W is polyconvex on $\mathrm{SL}(2)$.[7]

Now, assume that W is polyconvex on $\mathrm{SL}(2)$. Then there exists a convex function $P\colon \mathbb{R}^{2\times2} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$P(F, \det F) = \begin{cases} W(F) & : \ F \in \mathrm{SL}(2)\,, \\ \infty & : \ F \notin \mathrm{SL}(2)\,. \end{cases}$$

In particular, the function $\widetilde{P}\colon \mathbb{R}^{2\times2} \to \mathbb{R} \cup \{\infty\}$ with $\widetilde{P}(M) := P(M,1)$ for all $M \in \mathbb{R}^{2\times2}$ is convex, and

$$\widetilde{P}(F) = P(F,1) = P(F, \det F) = W(F)$$

for all $F \in \mathrm{SL}(2)$. $\square$

---

[7] Note that the minors of $F \in \mathbb{R}^{2\times2}$ are exactly the entries of F and the determinant of F.

A number of well-known criteria for the classical convexity on (subsets of) $\mathbb{R}^{n \times n}$ have been discussed in the literature. The following two lemmas, for example, provide equivalence criteria for the convexity on $\mathbb{R}^{n \times n}$ and $\mathrm{GL}^+(n)$, respectively, for functions which are *objective* as well as *isotropic* and only attain *finite values*. Note, however, that the mapping $\widetilde{P} \colon \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{\infty\}$ from Lemma B.1 does not need to have any of these three properties.

Lemma B.2 *([5, Theorem 5.1]). Let $\widetilde{P} \colon \mathbb{R}^{n \times n} \to \mathbb{R}$ be objective and isotropic with*

$$\widetilde{P}(M) = g(\lambda_1, \dots, \lambda_n)$$

for all $M \in \mathbb{R}^{n \times n}$ with singular values $\lambda_1, \dots, \lambda_n$. Then $\widetilde{P}$ is convex on $\mathbb{R}^{n \times n}$ if and only if g is convex on $\mathbb{R}_+^n$ and g is nondecreasing in each variable.

Lemma B.3 *([36, Theorem 3.1]). Let $\widetilde{P} \colon \mathrm{GL}^+(n) \to \mathbb{R}$ be objective and isotropic with*

$$\widetilde{P}(M) = g(\lambda_1, \dots, \lambda_n)$$

for all $M \in \mathbb{R}^{n \times n}$ with singular values $\lambda_1, \dots, \lambda_n$. Then $\widetilde{P}$ is convex on $\mathrm{GL}^+(n)$ (i.e. can be extended to a convex function on $\mathbb{R}^{n \times n}$) if and only if g is convex on $\mathbb{R}_+^n$ and g satisfies

$$g(\lambda_1, \dots, \lambda_n) \leq g(\lambda_1 + \beta, \dots, \lambda_n + \beta) \qquad \text{for all } \beta > 0 \, .$$

In order to show that

$$W : \mathrm{SL}(2) \to \mathbb{R}, \quad W(F) = \lambda_{\max} - \frac{1}{\lambda_{\max}}$$

is polyconvex on $\mathrm{SL}(2)$, we now consider the function

$$W_{\mathrm{ext}} : \mathbb{R}^{2 \times 2} \to \mathbb{R}, \quad W_{\mathrm{ext}} = \sqrt{\|F\|^2 - 2 \det(F)} \, .$$

Then W_{ext} is an extension of W to $\mathbb{R}^{2 \times 2}$, since

$$W(F) = \lambda_{\max} - \frac{1}{\lambda_{\max}} = |\lambda_1 - \lambda_2| = \sqrt{\lambda_1^2 + \lambda_2^2 - 2 \lambda_1 \lambda_2} = W_{\mathrm{ext}}(F)$$

for all $F \in \mathrm{SL}(2)$. In order to obtain the $\mathrm{SL}(2)$-polyconvexity of W from Lemma B.1, it therefore remains to show that W_{ext} is convex. This was first observed by Rosakis and Simpson [37, Lemma 3.1]: since

$$
\begin{aligned}
W_{\mathrm{ext}}(F) = \sqrt{\|F\|^2 - 2 \det F} &= \sqrt{F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2 - 2 \left(F_{11} F_{22} - F_{12} F_{21}\right)} \\
&= \sqrt{(F_{11} - F_{22})^2 + (F_{12} + F_{21})^2} \\
&= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} F_{11} - F_{22} & F_{12} + F_{21} \\ F_{12} + F_{21} & F_{22} - F_{11} \end{pmatrix} \right\| \\
&= \frac{\sqrt{2}}{2} \left\| \begin{pmatrix} 2\, F_{11} & F_{12} + F_{21} \\ F_{12} + F_{21} & 2\, F_{22} \end{pmatrix} - \begin{pmatrix} F_{11} + F_{22} & 0 \\ 0 & F_{11} + F_{22} \end{pmatrix} \right\| \\
&= \sqrt{2} \, \| \mathrm{sym}\, F - \tfrac{\mathrm{tr}\, F}{2} \cdot \mathbb{1} \| \; = \; \sqrt{2} \, \| \mathrm{dev}_2 \, \mathrm{sym}\, F \| \, ,
\end{aligned}
$$

the function W_{ext} is convex as the composition of the convex function $X \mapsto \sqrt{2}\,\|X\|$ and the linear mapping $F \mapsto \text{dev}_2 \, \text{sym} \, F$.

Note carefully that a real valued function W on $\mathbb{R}^{2 \times 2}$ can be represented in terms of the singular values of the argument if and only if

$$W(Q_1 \, F \, Q_2) = W(F) \quad \text{for all } F \in \text{GL}^+(2) \text{ and all } Q_1, Q_2 \in \text{O}(2), \tag{B.1}$$

where $\text{O}(2) = \{X \in \mathbb{R}^{2 \times 2} \,|\, X^T X = \mathbb{1}\}$ denotes the orthogonal group. While the restriction of the function W_{ext} to $\text{GL}^+(2)$ is objective and isotropic in the sense of Equation (2.1), it does not satisfy Equation (B.1), since in general $W_{\text{ext}}(QX) \neq W_{\text{ext}}(X) \neq W_{\text{ext}}(XQ)$ for $Q \in \text{O}(2) \setminus \text{SO}(2)$. In particular, W_{ext} cannot be represented in terms of the singular values of the argument on all of $\mathbb{R}^{2 \times 2}$.[8] Thus neither Lemma B.2 nor Lemma B.3 is applicable to W_{ext}.

References

[1] R.C. Abeyaratne, Discontinuous deformation gradients in plane finite elastostatics of incompressible materials, J. Elast. 10 (3) (1980) 255–293.

[2] J.J. Alibert, B. Dacorogna, An example of a quasiconvex function that is not polyconvex in two dimensions, Arch. Ration. Mech. Anal. 117 (2) (1992) 155–166.

[3] A.N. Annaidh, M. Destrade, M.D. Gilchrist, J.G. Murphy, Deficiencies in numerical models of anisotropic nonlinearly elastic materials, Biomech. Model. Mechanobiol. 12 (4) (2012) 781–791.

[4] G. Aubert, Necessary and sufficient conditions for isotropic rank-one convex functions in dimension 2, J. Elast. 39 (1) (1995) 31–46.

[5] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (4) (1976) 337–403.

[6] J.M. Ball, Constitutive inequalities and existence theorems in nonlinear elastostatics, in: R.J. Knops (Ed.), Herriot Watt Symposion: Nonlinear Analysis and Mechanics, vol. 1, Pitman, London, 1977, pp. 187–238.

[7] J.M. Ball, F. Murat, $W^{1,p}$-quasiconvexity and variational problems for multiple integrals, J. Funct. Anal. 58 (1984) 225–253.

[8] N. Chaudhuri, S. Müller, Rank-one convexity implies quasi-convexity on certain hypersurfaces, Proc. R. Soc. Edinb., Sect. A 133 (06) (2003) 1263–1272.

[9] S. Conti, Quasiconvex functions incorporating volumetric constraints are rank-one convex, J. Math. Pures Appl. 90 (1) (2008) 15–30.

[10] B. Dacorogna, Necessary and sufficient conditions for strong ellipticity of isotropic functions in any dimension, Discrete Contin. Dyn. Syst. 1 (2) (2001) 257–263.

[11] B. Dacorogna, Direct Methods in the Calculus of Variations, 2nd edition, Applied Mathematical Sciences, vol. 78, Springer, Berlin, 2008.

[12] P.J. Davies, A simple derivation of necessary and sufficient conditions for the strong ellipticity of isotropic hyperelastic materials in plane strain, J. Elast. 26 (3) (1991) 291–296.

[13] J.E. Dunn, R. Fosdick, Y. Zhang, Rank 1 convexity for a class of incompressible elastic materials, in: Rational Continua, Classical and New, Springer, 2003, pp. 89–96.

[14] W. Edelstein, R. Fosdick, A note on non-uniqueness in linear elasticity theory, Z. Angew. Math. Phys. 19 (6) (1968) 906–912.

[15] E. Ernst, Ellipticity loss in isotropic elasticity, J. Elast. 51 (3) (1998) 203–211.

[16] R. Fosdick, M.D. Piccioni, G. Puglisi, A note on uniqueness in linear elastostatics, J. Elast. 88 (1) (2007) 79–86.

[17] C. Horgan, J. Murphy, Constitutive modeling for moderate deformations of slightly compressible rubber, J. Rheol. 53 (2009) 153.

[18] J.K. Knowles, E. Sternberg, On the ellipticity of the equations of nonlinear elastostatics for a special material, J. Elast. 5 (3–4) (1975) 341–361.

[19] J.K. Knowles, E. Sternberg, On the failure of ellipticity of the equations for finite elastostatic plane strain, Arch. Ration. Mech. Anal. 63 (4) (1976) 321–336.

[20] P. Marcellini, Quasiconvex quadratic forms in two dimensions, Appl. Math. Optim. 11 (1) (1984) 183–189.

[21] R.J. Martin, I.D. Ghiba, P. Neff, Rank-one convexity implies polyconvexity for isotropic, objective and isochoric elastic energies in the two-dimensional case, Proc. R. Soc. Edinb. A 147 (3) (2015) 571–597, 2017.

[22] A. Mielke, Necessary and sufficient conditions for polyconvexity of isotropic functions, J. Convex Anal. 12 (2) (2005) 291–314.

[23] C. Mora-Corral, M. Strugaru, Necking in two-dimensional incompressible polyconvex materials: theoretical framework and numerical simulations, Q. J. Mech. Appl. Math. 70 (3) (2017) 249–271.

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[8] On $\text{GL}^+(n)$, W_{ext} can be represented as $W(F) = |\lambda_1 - \lambda_2|$. However, the function on all of $\mathbb{R}^{2 \times 2}$ with the same representation in terms of singular values λ_1, λ_2 is not convex.

[24] C.B. Morrey, Quasi-convexity and the lower semicontinuity of multiple integrals, Pac. J. Math. 2 (1) (1952) 25–53.
[25] S. Müller, Rank-one convexity implies quasiconvexity on diagonal matrices, Int. Math. Res. Not. 1999 (20) (1999) 1087–1095.
[26] S. Ndanou, N. Favrie, S. Gavrilyuk, Criterion of hyperbolicity in hyperelasticity in the case of the stored energy in separable form, J. Elast. 115 (1) (2014) 1–25.
[27] P. Neff, Critique of "Two-dimensional examples of rank-one convex functions that are not quasiconvex" by M.K. Benaouda and J.J. Telega, Ann. Pol. Math. 86 (2) (2005) 193–195.
[28] P. Neff, I.D. Ghiba, J. Lankeit, The exponentiated Hencky-logarithmic strain energy. Part I: constitutive issues and rank-one convexity, J. Elast. 121 (2015) 143–234.
[29] P. Neff, I.D. Ghiba, J. Lankeit, R.J. Martin, D.J. Steigmann, The exponentiated Hencky-logarithmic strain energy. Part II: coercivity, planar polyconvexity and existence of minimizers, Z. Angew. Math. Phys. 66 (2015) 1671–1693.
[30] R.W. Ogden, Non-Linear Elastic Deformations, 1st edition, Mathematics and Its Applications, Ellis Horwood, Chichester, 1983.
[31] P. Pedregal, Some remarks on quasiconvexity and rank-one convexity, Proc. R. Soc. Edinb., Sect. A 126 (05) (1996) 1055–1065.
[32] P. Pedregal, Some evidence in favor of Morrey's conjecture, preprint, arXiv:1406.7199, 2014.
[33] P. Pedregal, V. Sverak, A note on quasiconvexity and rank-one convexity for 2×2 matrices, J. Convex Anal. 5 (1998) 107–117.
[34] A. Raoult, Non-polyconvexity of the stored energy function of a St. Venant-Kirchhoff material, Apl. Mat. 31 (1986) 417–419.
[35] H. Richter, Das isotrope Elastizitätsgesetz, Z. Angew. Math. Mech. 28 (7–8) (1948) 205–209, available at https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/richter_isotrop_log.pdf.
[36] P. Rosakis, Characterization of convex isotropic functions, J. Elast. 49 (1997) 257–267.
[37] P. Rosakis, C.H. Simpson, On the relation between polyconvexity and rank-one convexity in nonlinear elasticity, J. Elast. 37 (2) (1994) 113–137.
[38] C. Sansour, On the physical assumptions underlying the volumetric-isochoric split and the case of anisotropy, Eur. J. Mech. A, Solids 27 (1) (2008) 28–39.
[39] J. Schröder, P. Neff, Poly, Quasi and Rank-One Convexity in Mechanics, CISM-Course Udine, Springer, 2009.
[40] D. Serre, Formes quadratiques et calcul des variations, J. Math. Pures Appl. 62 (9) (1983) 177–196.
[41] M. Šilhavý, The Mechanics and Thermomechanics of Continuous Media, Springer, Berlin, 1997.
[42] M. Šilhavý, Convexity conditions for rotationally invariant functions in two dimensions, in: Sequeira, et al. (Eds.), Applied Nonlinear Analysis, Kluwer Academic Publisher, New York, 1999.
[43] M. Šilhavý, On isotropic rank one convex functions, Proc. R. Soc. Edinb. 129 (1999) 1081–1105.
[44] M. Šilhavý, Rank 1 convex hulls of isotropic functions in dimension 2 by 2, Math. Bohem. 126 (2) (2001) 521–529.
[45] M. Šilhavỳ, Convexity conditions for rotationally invariant functions in two dimensions, in: Applied Nonlinear Analysis, Springer, 2002, pp. 513–530.
[46] M. Šilhavý, Monotonicity of rotationally invariant convex and rank 1 convex functions, Proc. R. Soc. Edinb. 132 (2002) 419–435.
[47] M. Šilhavý, An O(n) invariant rank 1 convex function that is not polyconvex, Theor. Appl. Mech. (Belgrad) 28 (2002) 325–336.
[48] M. Šilhavý, On SO(n)-invariant rank 1 convex functions, J. Elast. 71 (2003) 235–246.
[49] V. Sverak, Rank-one convexity does not imply quasiconvexity, Proc. R. Soc. Edinb. A 120 (1992) 185–189.
[50] F.J. Terpstra, Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung, Math. Ann. 116 (1) (1939) 166–180.
[51] L. Zee, E. Sternberg, Ordinary and strong ellipticity in the equilibrium theory of incompressible hyperelastic solids, Arch. Ration. Mech. Anal. 83 (1983) 53–90.
[52] L.M. Zubov, A.N. Rudev, Necessary and sufficient criteria for ellipticity of the equilibrium equations of a non-linearly elastic medium, J. Appl. Math. Mech. 59 (2) (1995) 197–208.
[53] L.M. Zubov, A.N. Rudev, A criterion for the strong ellipticity of the equilibrium equations of an isotropic nonlinearly elastic material, J. Appl. Math. Mech. 75 (2011) 432–446.