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Some Poincaré type inequalities for quadratic matrix fields

Sebastian Bauer[1], **Patrizio Neff**[1], **Dirk Pauly**[1,*], and **Gerhard Starke**[1]

[1] Universität Duisburg-Essen, Campus Essen, Fakultät für Mathematik, Thea-Leymann-Str. 9, 45141 Essen, Germany

We present some Poincaré type inequalities for quadratic matrix fields with applications e.g. in gradient plasticity or fluid dynamics. In particular, an application to the pseudostress-velocity formulation of the stationary Stokes problem is discussed.

1 Results

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ. Moreover, let $\gamma \neq \emptyset$ be a relatively open subset of Γ. The usual L^2-Sobolev spaces for the gradient, rotation and divergence with homogeneous scalar, tangential resp. normal trace on γ will be denoted by $H^1_\gamma(\Omega)$, $H_\gamma(\mathrm{curl}, \Omega)$, $H_\gamma(\mathrm{div}, \Omega)$, respectively. A matrix-valued function $T : \Omega \to \mathbb{R}^{3\times 3}$ belongs to the Sobolev space $H_\gamma(\mathrm{Curl}, \Omega)$ resp. $H_\gamma(\mathrm{Div}, \Omega)$, if its rows are elements of $H_\gamma(\mathrm{curl}, \Omega)$ resp. $H_\gamma(\mathrm{div}, \Omega)$, and the differential operators Curl and Div act row-wise. Moreover, we will use the standard matrix operations $\mathrm{sym}\, T = \frac{1}{2}(T + T^\top)$, $\mathrm{skew}\, T = \frac{1}{2}(T - T^\top)$ and $\mathrm{dev}\, T = T - \mathrm{tr}\, T/3 \cdot \mathbf{1}$, where $\mathbf{1}$ denotes the identity matrix. The standard L^2-norms for scalar, vector- or matrix-valued functions are denoted by $|\cdot|$. The following inequalities hold:

Theorem 1.1 *There exists a constant $c > 0$ such that for all $T \in H_\gamma(\mathrm{Curl}, \Omega)$*

$$c\,|T| \leq |\,\mathrm{dev}\,\mathrm{sym}\, T| + |\,\mathrm{Curl}\, T|.$$

See also our papers [7–11].

Proof. The proof follows in close lines our paper [10] and relies on three essential tools, namely • the Maxwell estimate (a Poincaré-type estimate for curl and div); • the Helmholtz decomposition; • Korn's first inequality. The only minor change is to prove a stronger version of Korn's first inequality:

$$\forall\, v \in H^1_\gamma(\Omega) \qquad c\,|\nabla v| \leq |\,\mathrm{dev}\,\mathrm{sym}\,\nabla v|.$$

The rest of the proof is identically to [10]. □

Theorem 1.2 *There exists a constant $c > 0$ such that for all $T \in H_\gamma(\mathrm{Div}, \Omega)$ the estimate*

$$c\,|T| \leq |\,\mathrm{dev}\, T| + |\,\mathrm{Div}\, T|$$

holds true. Especilly, for $S \in H_\gamma(\mathrm{Curl}, \Omega)$

$$c\,|\,\mathrm{Curl}\, S| \leq |\,\mathrm{dev}\,\mathrm{Curl}\, S|$$

holds, since $\mathrm{Curl}\, S \in H_\gamma(\mathrm{Div}, \Omega)$ is solenoidal.

Proof. Let $\tilde{\gamma} := \Gamma \setminus \overline{\gamma}$ be the complement of γ. Following [12], we first prove

$$\exists\, c > 0 \quad \forall\, f \in L^2(\Omega) \quad \exists\, v \in H^1_{\tilde{\gamma}}(\Omega) \qquad \mathrm{div}\, v = f, \quad |v| + |\nabla v| \leq c|f|.$$

Then, we utilize the idea of [1, Lemma 3.1] and obtain with some $v \in H^1_{\tilde{\gamma}}(\Omega)$ solving $\mathrm{div}\, v = \mathrm{tr}\, T$

$$\begin{aligned}|\,\mathrm{tr}\, T|^2 &= \langle \mathrm{tr}\, T, \mathrm{div}\, v\rangle = \langle \mathrm{tr}\, T, \mathrm{tr}\,\nabla v\rangle = \langle \mathrm{tr}\, T \cdot \mathbf{1}, \nabla v\rangle = 3\langle T, \nabla v\rangle - 3\langle \mathrm{dev}\, T, \nabla v\rangle \\ &= -3\langle \mathrm{Div}\, T, v\rangle - 3\langle \mathrm{dev}\, T, \nabla v\rangle \leq c\big(|\,\mathrm{dev}\, T| + |\,\mathrm{Div}\, T|\big)|\,\mathrm{tr}\, T|,\end{aligned}$$

which completes the proof, since it is sufficient to estimate $\mathrm{tr}\, T$. □

Corollary 1.3 *There exists a constant $c > 0$ such that for all $T \in H_\gamma(\mathrm{Curl}, \Omega)$*

$$c\,|T| \leq |\,\mathrm{dev}\,\mathrm{sym}\, T| + |\,\mathrm{dev}\,\mathrm{Curl}\, T|.$$

Proof. Since $S := \mathrm{Curl}\, T \in H_\gamma(\mathrm{Div}, \Omega)$ with $\mathrm{Div}\, S = 0$ the assertion follows immediately by a combination of Theorem 1.1 and Theorem 1.2. □

Remark 1.4 The results from Theorem 1.2 remain true in the L^p-setting with $1 < p < \infty$. They even extend to $T \in L^p(\Omega)$ only. Then $\mathrm{Div}\, T \in W^{-1,p}_\gamma(\Omega)$ and we have

$$c\,|T|_p \leq |\,\mathrm{dev}\, T|_p + |\,\mathrm{Div}\, T|_{-1,p,\gamma}.$$

* Corresponding author: Email dirk.pauly@uni-due.de, phone +00 49 201 183 7433, fax +00 49 201 183 4394

2 Application

Let us study the following first-order system formulation of the stationary Stokes equations: For some given vector field f in $L^2(\Omega)$ find a scalar function $p \in L^2(\Omega)$, a vector-valued function $u \in H^1_\gamma(\Omega)$ and a matrix-valued function $\sigma \in H_{\tilde{\gamma}}(\operatorname{Div}, \Omega)$, such that the system

$$\sigma - \mu \operatorname{sym} \nabla u + p \cdot \mathbf{1} = 0, \quad \operatorname{Div} \sigma = f, \quad \operatorname{div} u = 0$$

holds in Ω. This system is equivalent to

$$\operatorname{dev} \sigma - \mu \operatorname{sym} \nabla u = 0, \quad \operatorname{Div} \sigma = f,$$

where the pressure p has been eliminated and can be computed afterwards as $p = -\operatorname{tr} \sigma / 3$. For this first-order system, a least squares formulation based on minimizing the quadratic functional

$$|\operatorname{dev} \sigma - \mu \operatorname{sym} \nabla u|^2 + |\operatorname{Div} \sigma - f|^2$$

with respect to $(u, \sigma) \in H^1_\gamma(\Omega) \times H_{\tilde{\gamma}}(\operatorname{Div}, \Omega)$ was studied in [3, section 3.2]. The well-posedness of this least squares problem is shown based on a coercivity result of the form

$$|\operatorname{dev} \sigma - \mu \operatorname{sym} \nabla u|^2 + |\operatorname{Div} \sigma|^2 \geq c \left(|\operatorname{Div} \sigma|^2 + |\sigma|^2 + |\nabla u|^2 + |u|^2 \right) \tag{1}$$

to hold with a constant $c > 0$ for all $(u, \sigma) \in H^1_\gamma(\Omega) \times H_{\tilde{\gamma}}(\operatorname{Div}, \Omega)$. In order to obtain (1) and since the $H^1(\Omega)$-norm of u is controlled by $|\operatorname{sym} \nabla u|$ using Korn's first and Poincaré's inequalities, σ, more precisely $\operatorname{tr} \sigma$, needs to be controlled by the first-order system, i.e., the result of Theorem 1.2 is required. The inequality (1) is then proved in a way similar to the ellipticity proof in [4, Theorem 3.1] using

$$\begin{aligned} \langle \operatorname{dev} \sigma, \operatorname{sym} \nabla u \rangle = \langle \operatorname{sym} \operatorname{dev} \sigma, \nabla u \rangle = \langle \operatorname{sym} \sigma - \frac{1}{3} \operatorname{tr} \sigma \cdot \mathbf{1}, \nabla u \rangle &= \langle \sigma, \nabla u \rangle - \langle \operatorname{skew} \sigma, \nabla u \rangle - \frac{1}{3} \langle \operatorname{tr} \sigma, \operatorname{div} u \rangle \\ &= -\langle \operatorname{Div} \sigma, u \rangle - \langle \operatorname{skew} \sigma, \nabla u \rangle - \frac{1}{3} \langle \operatorname{tr} \sigma, \operatorname{div} u \rangle \end{aligned}$$

and

$$\begin{aligned} |\operatorname{skew} \sigma| &= |\operatorname{skew}(\operatorname{dev} \sigma - \mu \operatorname{sym} \nabla u)| \leq |\operatorname{dev} \sigma - \mu \operatorname{sym} \nabla u| \,, \\ \mu |\operatorname{div} u| &= |\operatorname{tr}(\operatorname{dev} \sigma - \mu \operatorname{sym} \nabla u)| \leq \sqrt{3} |\operatorname{dev} \sigma - \mu \operatorname{sym} \nabla u| \,. \end{aligned}$$

In [4], a result similar to Theorem 1.2 has been obtained indirectly by examination of the incompressible limit of a first-order system linear elasticity formulation.

A widely used result in the literature on mixed methods is for the inequality of Theorem 1.2 to hold for all $\sigma \in H(\operatorname{Div}, \Omega)$ which satisfy $\langle \operatorname{tr} \sigma, 1 \rangle = 0$. This result dates back to [1], see also [2, section IV.3] and is useful in the mixed framework where the boundary conditions on the normal stress is only treated weakly. Such mixed approaches have been analyzed recently for the stationary Stokes problem in [5] and [6].

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