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The Armstrong-Frederick Plasticity model with Cosserat effects

Krzysztof Chełmiński[1], **Patrizio Neff**[2], and **Sebastian Owczarek**[1,*]

[1] Faculty of Mathematics and Information Science, Warsaw University of Technology, Poland
[2] Fakultät für Mathematik, Universität Duisburg-Essen, Lehrstuhl für Nichtlineare Analysis und Modellierung, Germany

We propose an extension of equations formulated by Armstrong and Frederick which includes micropolar effects. We study existence of solutions to the quasistatic Armstrong-Frederick model with Cosserat effects which is still of non-monotone type. It was shown that the limit in the Yosida approximation process satisfies the energy inequality. The limit functions have a better regularity than it could be found in the literature, where the original Armstrong-Frederick model was studied.

1 Formulation of the problem

The original Armstrong-Frederick model describes the inelastic deformation process in metals. It is a modification of Melan-Prager model, which is well know in the literature and it can be treated as an approximation of the Prandtl-Reuss model. The modification is to add a nonlinear correction term to the equations for the beckstress. New term entails the L^∞- boundedness of the backstress. This new property is required in many applications, hence the Armstrong-Frederick model is very often used in practice. Unfortunately there is a drawback: This model is of non-monotone type and not of gradient type and the mathematical analysis is very difficult.

Here, we want to extension of equations proposed by P.J. Armstrong and C.O. Frederick in the article [1] to include micropolar effects. From the mechanical results for Cosserat plasticity in the papers [3] and [4] we conclude that we deal with the following initial-boundary value problem: we are looking for the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^3$, the microrotation matrix $A : \Omega \times [0,T] \to \mathfrak{so}(3)$ ($\mathfrak{so}(3)$ is the set of skew-symmetric 3×3 matrices) and the vector of internal variables $z = (\varepsilon^p, b) : \Omega \times [0,T] \to \mathcal{S}^3_{\mathrm{dev}} \times \mathcal{S}^3_{\mathrm{dev}}$ (ε^p describes the inelastic part of deformation, b is the so-called backstress tensor and the space $\mathcal{S}^3_{\mathrm{dev}}$ denotes the set of symmetric 3×3 - matrices with vanishing trace) satisfying the following system of equations

$$\begin{aligned}
\mathrm{div}_x T &= -f\,, \\
T &= 2\mu(\varepsilon(u) - \varepsilon^p) + 2\mu_c(\mathrm{skew}(\nabla_x u) - A) + \lambda \mathrm{tr}(\varepsilon(u) - \varepsilon^p)\mathbb{1}\,, \\
-l_c\,\Delta_x \mathrm{axl}\,(A) &= \mu_c\,\mathrm{axl}\,(\mathrm{skew}(\nabla_x u) - A)\,, \\
\varepsilon^p_t &\in \partial I_{K(b)}\Big(T_E\Big)\,, \\
T_E &= 2\mu(\varepsilon(u) - \varepsilon^p) + \lambda \mathrm{tr}(\varepsilon(u) - \varepsilon^p)\mathbb{1}, \\
b_t &= c\,\varepsilon^p_t - d\,|\varepsilon^p_t| b\,,
\end{aligned} \tag{1.1}$$

where $\varepsilon(u) = \mathrm{sym}(\nabla_x u)$ denotes the symmetric part of the gradient of the displacement. The above equations are studied for $x \in \Omega \subset \mathbb{R}^3$ and $t \in [0,T]$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ and t denotes the time.
The set of admissible stresses $K(b(x,t))$ is defined in the form $K(b) = \{T_E \in \mathcal{S}^3 \;:\; |\,\mathrm{dev}\,(T_E) - b| \le \sigma_{\mathrm{y}}\}$, where $\mathrm{dev}\,(T_E) = T_E - \frac{1}{3}\mathrm{tr}(T_E)\cdot \mathbb{1}$, σ_{y} is a material parameter (the yield limit) and $\mathbb{1}$ denotes the identity matrix. The function $I_{K(b)}$ is the indicator function of the set $K(b)$ and $\partial I_{K(b)}$ is the subgradient of the convex, proper, lower semicontinous function $I_{K(b)}$. $f : \Omega \times [0,T] \to \mathbb{R}^3$ is a given function describing density of the applied body forces. μ, λ are positive Lamé constants, $\mu_c > 0$ is the Cosserat couple modulus and $l_c > 0$ is a material parameter describing a length scale of the model due to the Cosserat effects. $c, d > 0$ are material constants. The operator "skew" denotes the skew-symmetric part of a 3×3 tensor. The operator $\mathrm{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ is the identification of the skew-symmetric matrix with vectors in $\mathbb{R}^3$.
The system (1.1) is considered with Dirichlet boundary conditions for the displacement and the microrotation:

$$u(x,t) = g_D(x,t), \quad A(x,t) = A_D(x,t) \qquad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t \ge 0 \tag{1.2}$$

Finally, we consider the system (1.1) with the following initial conditions

$$\varepsilon^p(x,0) = \varepsilon^{p,0}(x), \qquad b(x,0) = b^0(x). \tag{1.3}$$

* Corresponding author: e-mail s.owczarek@mini.pw.edu.pl

2 Main result

Let us consider convex set (the construction of this set appears in [2] and it will be used as set of test functions further on)

$$\mathcal{K}^* = \{(\mathrm{dev}\,(T_E), -\frac{1}{c}b) \in \mathcal{S}^3_{\mathrm{dev}} \times \mathcal{S}^3_{\mathrm{dev}} \,:\, |\,\mathrm{dev}\,(T_E) - b| + \frac{d}{2c}|b|^2 \leq \sigma_{\mathrm{y}}\}.$$

the construction of this set appears in

Definition 2.1 (solution concept − energy inequality)
Fix $T > 0$. Suppose that the given data satisfy some natural regularity, which are specified in [5]. We say that a vector $(u, T, A, \varepsilon^p, b) \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathcal{S}^3) \times H^2(\Omega; \mathfrak{so}(3)) \times L^2(\Omega; \mathcal{S}^3_{\mathrm{dev}}) \times L^\infty(\Omega; \mathcal{S}^3_{\mathrm{dev}}))$ solves the problem (1.1)-(1.3) if

$$(u_t, T_t, A_t, \varepsilon^p_t, b_t) \in L^2(0, T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^9) \times H^2(\Omega; \mathfrak{so}(3)) \times (L^2(\Omega; \mathcal{S}^3_{\mathrm{dev}}))^2),$$

the equations $(1.1)_1$ and $(1.1)_3$ are satisfied pointwise almost everywhere on $\Omega \times (0, T)$ and for all test functions $(\hat{T}_E, \hat{b}) \in L^2(0, T; L^2(\Omega; \mathcal{S}^3) \times L^2(\Omega; \mathcal{S}^3_{\mathrm{dev}}))$ such that

$$(\mathrm{dev}\,(\hat{T}_E), \hat{b}) \in \mathcal{K}^*, \qquad \mathrm{div}\,\hat{T}_E \in L^2(0, T; L^2(\Omega, \mathbb{R}^3))$$

the inequality

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathbb{C}^{-1}T_E(x,t)T_E(x,t)dx + \mu_c\int_\Omega |\mathrm{skew}(\nabla_x u(x,t)) - A(x,t)|^2dx \\
+\quad& 2l_c\int_\Omega |\nabla \mathrm{axl}(A(x,t))|^2dx + \frac{1}{2c}\int_\Omega |b(x,t)|^2dx \leq \frac{1}{2}\int_\Omega \mathbb{C}^{-1}T^0_E(x)T^0_E(x)dx \\
+\quad& \mu_c\int_\Omega |\mathrm{skew}(\nabla_x u(x,0)) - A(x,0)|^2dx + \frac{1}{2c}\int_\Omega |b(x,0)|^2dx + 2l_c\int_\Omega |\nabla \mathrm{axl}(A(x,0))|^2dx \\
+\quad& \int_0^t\int_\Omega u_t(x,\tau)f(x,\tau)dxd\tau + \int_0^t\int_\Omega u_t(x,\tau)\mathrm{div}\hat{T}_E(x,\tau)dxd\tau \\
+\quad& \int_0^t\int_{\partial\Omega} g_{D,t}(x,\tau)(T(x,\tau) - \hat{T}_E(x,\tau))\cdot n(x)dSd\tau + \int_0^t\int_\Omega \mathbb{C}^{-1}T_{E,t}(x,\tau)\hat{T}_E(x,\tau)dxd\tau \\
+\quad& \frac{1}{c}\int_0^t\int_\Omega b_t(x,\tau)\hat{b}(x,\tau)dxd\tau + 4l_c\int_0^t\int_{\partial\Omega} \nabla\mathrm{axl}(A(x,\tau))\cdot n\ \mathrm{axl}(A_{D,t}(x,\tau))dSd\tau
\end{aligned} \tag{2.4}$$

holds for all $t \in (0, T)$, where $T^0_E \in L^2(\Omega; \mathcal{S}^3)$ and $(u(0), A(0)) \in H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega; \mathfrak{so}(3))$ are unique solution of the problem $(1.1)_1 - (1.1)_3$ at the time equals zero and $\mathbb{C}^{-1} : \mathcal{S}^3 \to \mathcal{S}^3$ is a positive definite operator such that $\mathbb{C}^{-1}T_E = \varepsilon - \varepsilon^p$.

Theorem 2.2 (Main result)
Let us assume that the given data and initial data satisfy some natural regularity, which are specified in [5]. Then there exists a global in time solution (in the sense of Definition 2.1) of the system (1.1) with boundary condition (1.2) and initial condition (1.3).

The proof of Theorem 2.2 is divided into two parts. First, we use the Yosida Approximation to the maximal monotone part of the inelastic constitutive equation. Next, we pass to the limit to obtain a solution in the sense of Definition 2.1. The details can be found in the article [5].

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