Constructions for t-designs and s-resolvable t-designs

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Abstract

The purpose of the present paper is to introduce recursive methods for constructing simple t-designs, s-resolvable t-designs, and large sets of t-designs. The results turn out to be very effective for finding these objects. In particular, they reveal a fundamental property of the considered designs. Consequently, many new infinite series of simple t-designs, t-designs with s-resolutions and large sets of t-designs can be derived from the new constructions. For example, by starting with an important result of Teirlinck stating that for every natural number t and for all $N > 1$ there is a large set $LS[N](t, t+1, t+N \cdot \ell(t))$, where $\ell(t) = \prod_{i=1}^{t} \lambda(i) \cdot \lambda^*(i)$, $\lambda(t) = \mathrm{lcm}(\binom{t}{m} \mid m = 1, 2, \ldots, t)$ and $\lambda^*(t) = \mathrm{lcm}(1, 2, \ldots, t+1)$, we obtain the following statement. If $(t+2)$ is composite, then there is a large set $LS[N](t, t+2, t+1+N \cdot \ell(t))$ for all $N > 1$. If $(t+2)$ is prime, then there is an $LS[N](t, t+2, t+1+N \cdot \ell(t))$ for any N with $\gcd(t+2, N) = 1$.

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1 Introduction

Constructions of simple t-designs are a central theme in t-design theory. Several major approaches to the problem could be found in the literature such as the use of a possible automorphism group, the construction of large sets and the recursive construction methods. Automorphism group methods often need the use of computers to deal with huge combinatorial problems. For example, in 1982 Magliveras and Leavitt showed in a pioneer work the existence of the first non-trivial simple 6-designs by using groups [27]. Another way of dealing with methods by means of groups is to carry out group-theoretic arguments by using the knowledge of the group structures only. Actually, a great deal of simple t-designs are obtained by using groups, either through computer-based approaches [20, 21, 23, 24, 6, 7] or through group-theoretic arguments

[13, 3, 4, 5, 10, 11, 9, 16, 30]. In a seminal paper in 1987 [31] Teirlinck proved the existence of non-trivial simple t-designs for arbitrarily large t by constructing large sets. This achievement strongly motivates numerous researchers to develop further this part of t-design theory [18, 19, 1, 39, 26, 22, 25]. Recursive methods appear to be a vital element for the study of t-designs. Normally, statements obtained by recursive approaches are of general nature and do not limit on values of t [33, 34, 28, 35, 36, 37, 38].

In the present paper we introduce recursive methods for constructing t-designs, s-resolvable t-designs and large sets of t-designs. In fact, the results reveal a remarkable connection between the 'starting' and 'resulting' designs, and therefore, they are very useful. For example, they allow us to construct large sets of t-designs for arbitrarily large t which were not known before.

We recall some basic notation and definitions which are used in the remaining sections. A t-design, denoted by t-(v, k, λ), is a pair $(X, \mathcal{B})$, where X is a v-set of *points* and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, such that every t-subset of X is a subset of exactly λ blocks, and λ is called the *index* of the design. A t-design is called *simple* if no two blocks are identical, otherwise, it is called *non-simple*. A t-$(v, k, 1)$ design is called a *Steiner t-design*. The necessary conditions for the existence of a t-(v, k, λ) design are that

$$\lambda_i := \lambda \binom{v-i}{t-i} / \binom{k-i}{t-i} = \lambda \binom{v-i}{k-i} / \binom{v-t}{k-t}, \ \ 0 \le i \le t$$

are integers. Equivalently,

$$\lambda \binom{v-i}{t-i} \equiv 0 \left(\bmod \binom{k-i}{t-i} \right), \ \ 0 \le i \le t,$$

or

$$\lambda \binom{v-i}{k-i} \equiv 0 \left(\bmod \binom{v-t}{k-t} \right), \ \ 0 \le i \le t.$$

If these divisibility conditions for t, k , v , λ are satisfied, we may say for short that the parameters t-(v, k, λ) are *admissible*. The smallest positive integer λ for which these necessary conditions are satisfied is denoted by $\lambda_{\min}(t, k, v)$ or simply $\lambda_{\min}$. If $\mathcal{B}$ is the set of all k-subsets of X, then $(X, \mathcal{B})$ is a t-$(v, k, \lambda_{\max})$ design, called the *complete* design, where $\lambda_{\max} = \binom{v-t}{k-t}$. If we complement each block of a t-(v, k, λ) design with respect to the point set X, we get a t-$(v, v-k, \lambda^*)$ design with $\lambda^* = \lambda\binom{v-k}{t}/\binom{k}{t}$, hence we usually assume $k \le v/2$. Moreover, if there is a t-(v, k, λ) design, then $\lambda_{\min}$ divides λ; and we normally assume that $\lambda \le \lambda_{\max}/2$.

A t-(v, k, λ) design $(X, \mathcal{B})$ is called *s-resolvable*, for $1 \le s \le t-1$, if the block set $\mathcal{B}$ can be partitioned into $N \ge 2$ disjoint classes $\mathcal{B}_1, \ldots, \mathcal{B}_N$ such that each $(X, \mathcal{B}_i)$ is an s-(v, k, δ) design, for $i = 1, \ldots, N$. Each $\mathcal{B}_i$ is called an s-resolution class or simply a *resolution class*. It is also said that $(X, \mathcal{B})$ has an *s-resolution*; obviously $N = \lambda_s/\delta$. The necessary conditions for a t-(v, k, λ) design to be s-resolvable are that

$$N \mid \lambda_i, \ \ 0 \le i \le s.$$

If the complete k-$(v,k,1)$ design can be partitioned into N disjoint t-(v,k,λ) designs, for $t < k$, then it is said that there is a *large set* of t-(v,k,λ) designs, denoted by $LS[N](t,k,v)$, or $LS_\lambda(t,k,v)$. All designs in this paper are assumed to be simple. When the discussion concerns non-simple t-designs, this will be stated explicitly.

2 The first theorem

Let $(Z,\mathcal{B})$ be a simple t-(v,k,λ) design, where $Z=\{1,\dots,v\}$. Let $X = Z\cup\{v+1\} = \{1,\dots,v,v+1\}$. Define $Z_i = X\setminus\{i\}$ for $i=1,\dots,v+1$. In particular, $Z_{v+1}=Z$. Let $(Z_i,\mathcal{B}_i)$ be a copy of $(Z,\mathcal{B})$ defined on the point set $Z_i=\{1,\dots,i-1,i+1,\dots,v+1\}$ for $i=1,\dots,v+1$; note that $(Z_{v+1},\mathcal{B}_{v+1})=(Z,\mathcal{B})$. For a given $i=1,\dots,v$, the blocks of $\mathcal{B}_i$ are obtained from those of $\mathcal{B}_{v+1}$ as follows. Let $B=\{i_1,i_2,\dots,i_k\}\in\mathcal{B}_{v+1}$.

1. If $i\notin\{i_1,\dots,i_k\}$, then $B\in\mathcal{B}_i$.

2. If $i\in\{i_1,\dots,i_k\}$, say $i=i_j$, then $B_{i_j} := \{i_1,\dots,i_{j-1},v+1,i_{j+1},\dots,i_k\}\in\mathcal{B}_i$, i.e. B_{i_j} is derived from B by replacing i_j with $v+1$.

For each $i\in X$, define

$$\mathcal{D}_i = \{\{i\}\cup B \mid B\in\mathcal{B}_i\},$$

and

$$\mathcal{D} = \mathcal{D}_1\cup\mathcal{D}_2\cup\cdots\cup\mathcal{D}_{v+1}.$$

We claim that $(X,\mathcal{D})$ is a t-$(v+1,k+1,\Lambda^*)$ design with repeated blocks, where $\Lambda^* = \lambda\frac{v+1-t}{k+1-t}(k+1)$, and each block of $\mathcal{D}$ is repeated $(k+1)$ times. Let $T=\{i_1,\dots,i_t\}$ be a t-subset of X.

1. Consider $\mathcal{D}_i$, $i\notin\{i_1,\dots,i_t\}$. Then $T\subseteq Z_i$. Since $\mathcal{D}_i=\{\{i\}\cup B\mid B\in\mathcal{B}_i\}$, and $(Z_i,\mathcal{B}_i)$ is a t-(v,k,λ) design, there are λ blocks $D=\{i\}\cup B\in\mathcal{D}_i$ containing T. As there are $(v+1-t)$ such $\mathcal{D}_i$ with $i\notin\{i_1,\dots,i_t\}$, there are altogether $(v+1-t)\lambda$ blocks of this type in $\mathcal{D}$ containing T.

2. Consider $\mathcal{D}_{i_j}=\{\{i_j\}\cup B\mid B\in\mathcal{B}_{i_j}\}$ for the t remaining $\mathcal{D}_{i_j}$ with $i_j\in\{i_1,\dots,i_t\}$. Set $T_{i_j} := T\setminus\{i_j\}$. Then T_{i_j} appears $\lambda_{t-1}=\lambda\frac{v+1-t}{k+1-t}$ times in the blocks of $(Z_{i_j},\mathcal{B}_{i_j})$. Thus T appears $t\lambda_{t-1}$ times in the blocks of $\mathcal{D}_{i_1},\dots,\mathcal{D}_{i_t}$.

The two cases 1. and 2. give

$$\begin{aligned}\Lambda^* &= \lambda(v+1-t)+t\lambda_{t-1}\\ &= \lambda\frac{v+1-t}{k+1-t}(k+1),\end{aligned}$$

as desired.

Next we show that $(X,\mathcal{D})$ has repeated blocks and each block of $\mathcal{D}$ is repeated $(k+1)$ times. Without loss of generality consider a block $D=\{v+1\}\cup B\in\mathcal{D}_{v+1}$, where $B = \{i_1, \ldots, i_k\} \in \mathcal{B}_{v+1}$. Because $(Z_i, \mathcal{B}_i)$ is a copy of $(Z_{v+1}, \mathcal{B}_{v+1})$, $i = 1, \ldots, v$, we see that

$$\begin{aligned} B_{i_1} &= \{v+1, i_2, \ldots, i_k\} \in \mathcal{B}_{i_1} \\ B_{i_2} &= \{i_1, v+1, i_3, \ldots, i_k\} \in \mathcal{B}_{i_2} \\ \vdots &= \vdots \\ B_{i_k} &= \{i_1, i_2, \ldots, i_{k-1}, v+1\} \in \mathcal{B}_{i_k} \end{aligned}$$

It follows that the $(k+1)$ blocks

$$\begin{aligned} D_{i_1} &= \{i_1\} \cup B_{i_1} \in \mathcal{D}_{i_1} \\ D_{i_2} &= \{i_2\} \cup B_{i_2} \in \mathcal{D}_{i_2} \\ \vdots &= \vdots \\ D_{i_k} &= \{i_k\} \cup B_{i_k} \in \mathcal{D}_{i_k} \\ D &= \{v+1\} \cup B \in \mathcal{D}_{v+1} \end{aligned}$$

are identical. These blocks are the only repeated blocks of D. This can be seen as follows. If D' is a repeated block of D, then D' must be of the form $D' = \{j\} \cup B$ with $B \in \mathcal{B}_j$ for a $j \in \{i_1, \ldots, i_k, v+1\}$, say $j = i_h$. But since any two blocks in $\mathcal{D}_{j_h}$ are distinct, so $D' = D_{j_h}$, where D_{j_h} is one of the $(k+1)$ repeated blocks of D described above. Hence, $(X, \mathcal{D})$ is a non-simple t-$(v+1, k+1, \lambda\frac{v+1-t}{k+1-t}(k+1))$ design, in which any block is repeated $(k+1)$ times, which proves the claim. Define $\Lambda = \lambda\frac{v+1-t}{k+1-t}$. Now assume that the conditions

$$\Lambda\binom{v+1-i}{k+1-i} \equiv 0 \left(\bmod \binom{v+1-t}{k+1-t}\right), \quad 0 \le i \le t \tag{1}$$

are satisfied, then by removing the repeated blocks of $(X, \mathcal{D})$ we obtain a simple t-$(v+1, k+1, \Lambda)$ design $(X, \mathcal{C})$. Note that the conditions in (1) are the necessary conditions for the existence of $(X, \mathcal{C})$. A close look shows that the conditions in (1) are already satisfied for $1 \le i \le t$, because these cases coincide with the divisibility conditions

$$\lambda\binom{v-j}{k-j} \equiv 0 \left(\bmod \binom{v-t}{k-t}\right), \quad 0 \le j \le t-1 \tag{2}$$

for the t-(v, k, λ) design $(Z, \mathcal{B})$, which are assumed to be satisfied. Consequently, the conditions in (1) are fulfilled, if for $i = 0$ we have

$$\Lambda\binom{v+1}{k+1} \equiv 0 \left(\bmod \binom{v+1-t}{k+1-t}\right),$$

or equivalently,

$$\Lambda_0 = \Lambda\binom{v+1}{k+1}\Big/\binom{v+1-t}{k+1-t} = \lambda\frac{v+1-t}{k+1-t}\binom{v+1}{k+1}\Big/\binom{v+1-t}{k+1-t} = \lambda\binom{v+1}{k+1}\Big/\binom{v-t}{k-t}$$

is an integer. Hence we have proved the following theorem.

Theorem 2.1 *Assume that there exists a simple t-(v,k,λ) design.*

(i) Then there exists a non-simple t-$(v+1,k+1,\lambda\frac{v+1-t}{k+1-t}(k+1))$ design, in which every block is repeated $(k+1)$ times.

(ii) If $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ is an integer, then there exists a simple t-$(v+1,k+1,\lambda\frac{v+1-t}{k+1-t})$ design.

In Theorem 2.1, if $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ is an integer, then we simply say that the parameters t-$(v+1,k+1,\Lambda)$ with $\Lambda=\lambda\frac{v+1-t}{k+1-t}$ are admissible. Equivalently, Λ is a multiple of $\lambda_{\min}(t,k+1,v+1)$.

Remark 2.1 It is worth mentioning the special case with $v=2k+1$ of Theorem 2.1. Suppose that there is a t-$(2k+1,k,\lambda)$ design. Obviously, $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ is integral in this case, since $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}=\lambda\binom{2k+2}{k+1}/\binom{2k+1-t}{k-t}=\frac{2k+2}{k+1}\lambda\binom{2k+1}{k}/\binom{2k+1-t}{k-t}=2\lambda_0$. Thus the condition in Theorem 2.1 (ii) is always satisfied. Therefore we obtain a t-$(2k+2,k+1,\lambda\frac{2k+2-t}{k+1-t})$ design. The result for this special case can be found in [34], or [37], however it is proven by a different method.

The following examples illustrate Theorem 2.1.

- Consider a known 6-$(30,10,\lambda)$ design, with $\lambda=m42$, $1\le m\le 126$ [14]. Then we have $\Lambda=\lambda\frac{v+1-t}{k+1-t}=m\cdot 42\cdot 5$. Thus the parameters 6-$(31,11,\Lambda)$ are admissible, if and only if Λ is a multiple of $\lambda_{\min}(6,11,31)=462=42\cdot 11$. Since $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}=m\cdot 31\cdot 15\cdot 29\cdot 21\cdot 13/11$, the integral condition for $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ becomes $11\mid m$. There are 27 known values of m, of which 25 with $11\nmid m$, i.e. $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ is not an integer. The other two $m=99,121$ with $11\mid m$ give rise to 6-$(31,11,45\cdot 462)$, and 6-$(31,11,55\cdot 462)$ designs.

- By starting with a known 7-$(34,10,m15)$ design for any $m\in\{$ 35, 40, 43, 44, 47, 48, 49, 52, 55, 56, 59, 60, 63, 64, 67, 68, 71, 72, 75, 76, 79, 83, 84, 87, 88, 91, 92, 95, 96 $\}$ [14], we find that $\Lambda=\lambda\frac{v+1-t}{k+1-t}=m15\cdot 7=m105$. Since $\lambda_{\min}(7,11,35)=105$, the parameters 7-$(35,11,m105)$ are admissible. Hence we get a simple 7-$(35,11,m105)$ design for all these values of m.

  If we repeat applying Theorem 2.1 to 7-$(35,11,m105)$ designs, we find that $\Lambda=\lambda\frac{v+1-t}{k+1-t}=m29\cdot 21$. Since $\lambda_{\min}(7,12,36)=21$, the parameters 7-$(36,12,m29\cdot 21)$ are admissible. Hence we get a simple 7-$(36,12,m29\cdot 21)$ design for all the values of m above.

3 The second theorem

We keep the notation from the previous section. Now assume that the t-(v,k,λ) design $(Z,\mathcal{B})$ is s-resolvable. Let s-(v,k,δ_s) be the parameters of the designs in the resolution of $(Z,\mathcal{B})$, and let N be the number of resolution classes. Thus, the block set $\mathcal{B}$ can be written as $\mathcal{B} = \mathcal{B}^1 \cup \cdots \cup \mathcal{B}^N$, which is a partition of $\mathcal{B}$ into subsets $\mathcal{B}^j$ such that $(Z, \mathcal{B}^j)$ is an s-(v, k, δ_s) design for $j = 1, \ldots, N$. By Theorem 2.1 $(X, \mathcal{D})$ is a non-simple t-$(v+1, k+1, \Lambda(k+1))$ design with $\Lambda = \lambda\frac{v+1-t}{k+1-t}$, in which any block is repeated $(k+1)$ times. In particular, applying Theorem 2.1 to $(Z, \mathcal{B}^j)$, $j = 1, \ldots, N$, yields a non-simple s-$(v+1, k+1, \Delta_s(k+1))$ design $(X, \mathcal{D}^j)$ with $\Delta_s = \delta_s\frac{v+1-s}{k+1-s}$, and any block is repeated $(k+1)$ times. It follows that there is a partition of $(X, \mathcal{D})$ into N disjoint non-simple s-designs $(X, \mathcal{D}^1), \ldots, (X, \mathcal{D}^N)$. Now assume that $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ is an integer. Again by Theorem 2.1, if the divisibility conditions

$$\Delta_s\binom{v+1-i}{k+1-i} \;\equiv\; 0 \;\left(\bmod \binom{v+1-s}{k+1-s}\right), \;\; 0 \le i \le s \tag{3}$$

are satisfied, then by removing the repeated blocks of $(X, \mathcal{D}^j)$ we obtain a simple s-$(v+1, k+1, \delta_s\frac{v+1-s}{k+1-s})$ design $(X, \mathcal{C}^j)$ for $j = 1, \ldots, N$. Thus, in this way, the non-simple design $(X, \mathcal{D})$ will yield a simple t-$(v+1, k+1, \Lambda)$ design $(X, \mathcal{C})$, which is resolvable into s-designs $(X, \mathcal{C}^j)$, $j = 1, \ldots, N$. Moreover, the s-resolvability of $(Z, \mathcal{B})$ implies that $N\delta_i = \lambda_i$ for $0 \le i \le s$, or equivalently,

$$N \mid \lambda_i \;\text{ for }\; 0 \le i \le s.$$

Thus the s-resolvability conditions in (3) for $(X, \mathcal{C})$ are equivalent to

$$N \mid \Lambda_i \;\text{ for }\; 0 \le i \le s.$$

Observe that the conditions $N \mid \Lambda_i$ for $1 \le i \le s$ are met, since they are identical to $N \mid \lambda_j$ for $0 \le j \le s-1$, which are already satisfied by the assumption. Thus, the conditions $N \mid \Lambda_i$ for $0 \le i \le s$ are satisfied, if $N \mid \Lambda_0$, which is $N \mid \lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$. Hence, we have proved the following result.

Theorem 3.1 *Assume that there exists a simple s-resolvable t-(v, k, λ) design with N resolution classes. If $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ is an integer and N divides $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$, then there exists a simple s-resolvable t-$(v+1, k+1, \lambda\frac{v+1-t}{k+1-t})$ design.*

The most important consequence of Theorem 3.1 is the following corollary.

Corollary 3.2 *If there exists a large set $LS[N](t, k, v)$ such that N divides $\binom{v+1}{k+1}$, then there exists a large set $LS[N](t, k+1, v+1)$.*

Proof. An $LS[N](t, k, v)$ is a partition of the complete k-$(v, k, 1)$ design into N disjoint t-designs with $t < k$. The complete design is also a t-$(v, k, \binom{v-t}{k-t})$ design. Therefore, $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t} = \binom{v+1}{k+1}$ is an integer. Moreover, the resulting t-$(v+1, k+1, \lambda\frac{v+1-t}{k+1-t})$ design becomes a simple t-$(v+1, k+1, \binom{v+1-t}{k+1-t})$ design, which is the complete $(k+1)$-$(v+1, k+1, 1)$ design. Hence, the corollary follows. $\square$

Remark 3.1 The following recursive construction for large sets is known: "If an $LS[N](t, k, v)$ and an $LS[N](t, k+1, v)$ exist, then there exists an $LS[N](t, k+1, v+1)$", see [2] and also [37]. In fact, Corollary 3.2 is the most general result we can get from a recursive construction of large sets, since no assumption is required, except for the necessary divisibility conditions. Actually, Corollary 3.2 can be stated as follows.

If there exists an $LS[N](t,k,v)$, then there exists an $LS[N](t,k+1,v+1)$, provided that the parameters of the latter are admissible.

It is appropriate to include a remark on the conditions in Theorem 3.1. The first condition requires that $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$ is an integer and the second that N divides $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$. Thus, if the first condition is not satisfied, then neither is the second. However, if the first condition is fulfilled, then the second does not need to be met. The following example clarifies these cases and also shows an iterated use of Theorem 3.1.

Consider a Steiner 5-$(84,6,1)$ design whose blocks are a union of 18 long block orbits of $\mathrm{PSL}_2(83)$, (i.e. orbits of length $|\mathrm{PSL}_2(83)|$). The design is thus 3-resolvable with $N = 18$ resolution classes [8]. Applying Theorem 3.1 yields a 3-resolvable 5-$(85,7,4\cdot 10)$ design, since both conditions are satisfied. By repeated application to this resulting 5-$(85,7,4\cdot 10)$ design, we again get a 3-resolvable 5-$(86,8,54\cdot 20)$ design. Now, by applying Theorem 3.1 to the 5-$(86,8,54\cdot 20)$ design, we find that $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t} = 87\cdot 43\cdot 85\cdot 7\cdot 83\cdot 41$ is an integer. Hence, we get a 5-$(87,9,738\cdot 30)$ design. However, since $N = 18$ does not divide $87\cdot 43\cdot 85\cdot 7\cdot 83\cdot 41$, the latter is not 3-resolvable.

Here is an example for Corollary 3.2. Consider an $LS_1(2,4,16) = LS[91](2,4,16)$ constructed by Mathon [29]. Since $91 \mid \binom{17}{5}$, Corollary 3.2 gives an $LS_5(2,5,17) = LS[91](2,5,17)$. Note that $5 = \lambda_{\min}(2,5,17)$. Again if we apply Corollary 3.2 to $LS[91](2,5,17)$, we find that $91 \mid \binom{18}{6}$. Hence there also exists an $LS[91](2,6,18)$. It appears that $LS[91](2,5,17)$ and $LS[91](2,6,18)$ are unknown to date.

We should note that $LS[91](2,4,16)$ is the second known large set of Steiner t-$(v,k,1)$ designs with $t \geq 2$ and $k \geq 4$; the first one is $LS_1(2,4,13) = LS[55](2,4,13)$ [15, 17]. However, there is no $LS[55](2,5,14)$, since these parameters are not admissible, in particular $55 \nmid \binom{14}{5}$.

Again we remark that the conditions of Theorems 2.1, 3.1 and Corollary 3.2 are simply the necessary divisibility conditions required for a t-design, or a t-design with resolution to exist. Generally, these necessary conditions are implicitly assumed, when the existence or the resolvability of a design is concerned. Hence, in this sense, the above theorems and corollary actually prove an intrinsic connection between the 'starting' and 'resulting' designs.

4 Applications

A few examples in the preceding sections have already suggested that the methods are useful. Actually, if applied fully, the results will produce infinitely many new t-designs, t-designs with s-resolutions, and large sets of t-designs. In this section, however, we limit our attention to some infinite series of t-designs for $t \geq 4$, which are derived from Theorems 2.1, 3.1 and Corollary 3.2, as an assertion of their unexpected strength.

4.1 t-designs and s-resolvable t-designs

1. We begin with a 4-$(2^n+1,5,5)$ design, for n odd, $n \geq 5$, constructed by Alltop [3]. It is easily checked that $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t} = 5\binom{2^n+2}{6}/\binom{2^n-3}{1}$ is an integer, because n is odd. Hence, by Theorem 2.1(ii), there is a 4-$(2^n+2,6,5(2^{n-1}-1))$ design. Moreover, Alltop's design is 3-resolvable with $N=(2^n-2)/6$ resolution classes. It can also be verified that N divides $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$, thus, by Theorem 3.1 the resulting 4-$(2^n+2,6,5(2^{n-1}-1))$ design is 3-resolvable with N resolution classes.

2. When starting with a 3-resolvable 4-$(2^n+1,6,10)$ design, for n odd, $n \geq 5$, with $N=(2^n-2)/6$ resolution classes, constructed by Bierbrauer [10], we find that $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t} = 10\binom{2^n+2}{7}/\binom{2^n-3}{2}$ is an integer if $n \equiv 0,1 \pmod 3$, hence, by Theorem 2.1 there is a 4-$(2^n+2,7,\frac{20}{3}(2^{n-1}-1))$ design. Moreover, $N \mid \lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$, if $n \equiv 0 \pmod 3$. In this case, the resulting design is 3-resolvable by Theorem 3.1.

3. Consider a 3-resolvable 4-$(2^n+1,9,84)$ design, for $\gcd(n,6)=1$, $n \geq 5$, with $N=(2^n-2)/6$ resolution classes, constructed by Bierbrauer [11]. It is straightforward to verify that $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t} = 84\binom{2^n+2}{10}/\binom{2^n-3}{5}$ is an integer. Moreover $N \mid \lambda\binom{v+1}{k+1}/\binom{v-t}{k-t}$, if $n \equiv 3 \pmod 4$. Hence, by Theorem 3.1, we obtain a 3-resolvable 4-$(2^n+2,10,28(2^{n-1}-1))$ design for $\gcd(n,6)=1$, $n \equiv 3 \pmod 4$, $n \geq 5$.

We summarize these results in the following theorem.

Theorem 4.1 *Let n be an odd integer such that $n \geq 5$. Then we have the following.*

(i) There exists a 3-resolvable 4-$(2^n+2,6,5(2^{n-1}-1))$ design with $N=(2^n-2)/6$ resolution classes.

(ii) If $n \equiv 0,1 \pmod 3$, then there exists a 4-$(2^n+2,7,\frac{20}{3}(2^{n-1}-1))$ design. Moreover if $n \equiv 0 \pmod 3$, the design is 3-resolvable with $N=(2^n-2)/6$ resolution classes.

(iii) If $\gcd(n,6)=1$, then there exists a 4-$(2^n+2,10,28(2^{n-1}-1))$ design. Furthermore, if $n \equiv 3 \pmod 4$, the design is 3-resolvable with $N=(2^n-2)/6$ resolution classes.

Consider a further example of an infinite class of 5-designs. By Theorem 5.1 [38] there exists a simple 5-$(5+28m,6,h4m)$ design for $h=1,2,3$ and $m \geq 1$. By applying Theorem 2.1, we find that $\lambda\binom{v+1}{k+1}/\binom{v-t}{k-t} = 4hm\binom{6+28m}{7}/\binom{28m}{1} = h4m(6+28m)\cdots(1+28m)/7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2$ is an integer if and only if $7 \mid m$, and in this case we get a 5-$(6+28m,7,h2m(1+28m))$ design. Hence the next result follows.

Theorem 4.2 *There exists a simple 5-$(6+28m,7,h2m(1+28m))$ design for $h=1,2,3$ and for any positive integer m such that $7 \mid m$.*

4.2 Large sets of t-designs

In [32] Teirlinck constructed infinite series of large sets of t-$(t + N \cdot \ell(t), t + 1, \ell(t))$ designs for every natural number t and for all $N > 1$, where $\ell(t) = \prod_{i=1}^{t} \lambda(i) \cdot \lambda^*(i)$, $\lambda(t) = \mathrm{lcm}(\binom{t}{m} \mid m = 1, 2, \ldots, t)$ and $\lambda^*(t) = \mathrm{lcm}(1, 2, \ldots, t + 1)$. Equivalently, an $LS[N](t, t + 1, t + N \cdot \ell(t))$ exists. This is an important result in t-design theory.

Consider an $LS[N](t, t+1, t+N \cdot \ell(t))$. By applying Corollary 3.2 we will obtain an $LS[N](t, t + 2, t + 1 + N \cdot \ell(t))$, if $\binom{t+1+N\cdot\ell(t)}{t+2}/N$ is an integer.

Now

$$\frac{\binom{t+1+N\cdot\ell(t)}{t+2}}{N} = \frac{(t + 1 + N \cdot \ell(t))(t + N \cdot \ell(t))(t - 1 + N \cdot \ell(t)) \cdots (1 + N \cdot \ell(t))N \cdot \ell(t)}{(t + 2)(t + 1)t(t - 1) \cdots 2 \cdot 1 \cdot N}.$$

First of all note that $(t + 1)!N$ divides $N \cdot \ell(t)$. More precisely, $t!N \mid N \cdot \lambda(t)\lambda(t - 1) \cdots \lambda(2)\lambda(1)$ and $(t + 1) \mid \lambda^*(t)$, so $(t + 1)!N \mid N \cdot \lambda(t)\lambda(t - 1) \cdots \lambda(2)\lambda(1) \cdot \lambda^*(t)$.

- If $t + 2$ is composite, we write $t + 2 = a \cdot b$ with $2 \leq a, b \leq t$. Then $a \cdot b \mid \lambda^*(t-1)\cdot\lambda^*(t)$. Moreover, since $\gcd(t+2, t+1) = \gcd(a, t+1) = \gcd(b, t+1) = 1$, we have $a\cdot b\cdot(t+1) \mid \lambda^*(t-1)\cdot\lambda^*(t)$. So, $(t+2)(t+1)!N \mid \ell(t)$. Thus $\binom{t+1+N\cdot\ell(t)}{t+2}/N$ is an integer for all $N > 1$.

- If $t + 2$ is prime, then $\gcd(t + 2, \ell(t)) = 1$. We have either $(t + 2) \mid N$ or $(t + 2) \nmid N$. If $(t + 2) \mid N$, then $(t + 2) \nmid (t + 1 + N \cdot \ell(t))(t + N \cdot \ell(t))(t - 1 + N \cdot \ell(t)) \cdots (1 + N \cdot \ell(t))$, therefore $\binom{t+1+N\cdot\ell(t)}{t+2}/N$ is not an integer. If $(t + 2) \nmid N$, then $(t + 2) \mid (t + 1 + N \cdot \ell(t))(t + N \cdot \ell(t))(t - 1 + N \cdot \ell(t)) \cdots (1 + N \cdot \ell(t))$. Thus $\binom{t+1+N\cdot\ell(t)}{t+2}/N$ is an integer for any N with $(t + 2) \nmid N$.

Hence, we have the following result.

Theorem 4.3 *For every natural number t let $\lambda(t) = \mathrm{lcm}(\binom{t}{m} \mid m = 1, 2, \ldots, t)$, $\lambda^*(t) = \mathrm{lcm}(1, 2, \ldots, t + 1)$ and $\ell(t) = \prod_{i=1}^{t} \lambda(i) \cdot \lambda^*(i)$. Then*

(i) if $(t + 2)$ is composite, there exists an $LS[N](t, t + 2, t + 1 + N \cdot \ell(t))$ for every $N \geq 1$,

(ii) if $(t + 2)$ is prime, there exists an $LS[N](t, t + 2, t + 1 + N \cdot \ell(t))$ for any $N \geq 1$ with $(t + 2) \nmid N$.

Note that if we emphasize t-designs in place of large sets of t-designs, then Theorem 4.3 provides the following corollary.

Corollary 4.4 *For every natural number t let $\lambda(t) = \mathrm{lcm}(\binom{t}{m} \mid m = 1, 2, \ldots, t)$, $\lambda^*(t) = \mathrm{lcm}(1, 2, \ldots, t + 1)$ and $\ell(t) = \prod_{i=1}^{t} \lambda(i) \cdot \lambda^*(i)$. Then there exists a t-$(t + 1 + N \cdot \ell(t), t + 2, \ell(t)(1 + N \cdot \ell(t))/2)$ design for all $N > 1$, if $(t + 2)$ is composite; and for any N with $(t + 2) \nmid N$, if $(t + 2)$ is prime.*

We want to apply Corollary 3.2 to $LS[N](t, t+2, t+1+N\cdot\ell(t))$ of Theorems 4.3 anew. For the sake of simplicity we assume that $t \geq 2$, and check the conditions for which $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is an integer. For $2 \leq t \leq 5$ it is easy to verify these conditions directly from the values of $\ell(t)$ and N and the result is as follows.

(i) If $(t+3)$ and $(t+2)$ are both composite, then $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is an integer for all $N \geq 1$.

(ii) If $(t+2)$ is prime, then $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is an integer for all $N \geq 1$ with $(t+2) \nmid N$.

(iii) If $(t+3)$ is prime, then $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is an integer for all $N \geq 1$ with $(t+3) \nmid N$.

So, we have to prove the validity of (i), (ii), (iii) for all $t \geq 6$. We begin with a remark. If $(t+3)$ and $(t+2)$ are composite for $t \geq 6$, and $(t+3) = A\cdot B$ and $(t+2) = a\cdot b$ are non-trivial factorizations, then $2 \leq A, B, a, b \leq t-2$. Now

$$\frac{\binom{t+2+N\cdot\ell(t)}{t+3}}{N} = \frac{(t+2+N\cdot\ell(t))(t+1+N\cdot\ell(t))(t+N\cdot\ell(t))\cdots(1+N\cdot\ell(t))N\cdot\ell(t)}{(t+3)(t+2)(t+1)t(t-1)\cdots 2\cdot 1\cdot N}.$$

- Assume that $(t+3)$ and $(t+2)$ are both composite and $(t+3) = A\cdot B$, $(t+2) = a\cdot b$, are their non-trivial factorizations with $2 \leq A, B, a, b \leq t-2$. We have $t! \mid \lambda(1)\cdot\lambda(2)\cdot\ldots\cdot\lambda(t)$ and $(t+1) \mid \lambda^*(t)$. Since $\gcd(t+2, t+1) = \gcd(a, t+1) = \gcd(b, t+1) = 1$, it follows that $b\cdot(t+1) \mid \lambda^*(t)$. Again since $\gcd(t+3, t+2) = \gcd(A, t+2) = \gcd(B, t+2) = 1$, we have $A\cdot B\cdot a \mid \lambda^*(t-2)\cdot\lambda^*(t-1)$. So, $A\cdot B\cdot a\cdot b\cdot(t+1) \mid \lambda^*(t-2)\cdot\lambda^*(t-1)\lambda^*(t)$. Thus $(t+3)(t+2)(t+1)t! \mid \lambda(1)\cdot\lambda(2)\cdots\lambda(t)\cdot\lambda^*(t-2)\cdot\lambda^*(t-1)\lambda^*(t)$ and therefore $(t+3)(t+2)(t+1)t! \mid \ell(t)$. Hence $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is an integer for all $N \geq 2$.

- Assume that $(t+2)$ is prime. Then $(t+3)$ is composite. Let $(t+3) = A\cdot B$ with $2 \leq A, B \leq t-2$. Since $(t+2)$ is prime, we have $(t+2) \nmid N$ by Theorem 4.1, and $(t+2) \mid (t+1+N\cdot\ell(t))(t+N\cdot\ell(t))\cdots(1+N\cdot\ell(t))$. Since $(t+3)(t+1) = A\cdot B\cdot(t+1)$, we have $A\cdot B\cdot(t+1) \mid \lambda^*(t-2)\cdot\lambda^*(t-1)\lambda^*(t)$. This implies that $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is an integer.

- Assume that $(t+3)$ is prime. Then $(t+2)$ is composite. If $(t+3) \mid N$, then $(t+3) \nmid (t+2+N\cdot\ell(t))(t+1+N\cdot\ell(t))(t+N\cdot\ell(t))\cdots(1+N\cdot\ell(t))$; so $(t+3)(t+2)(t+1)!N \nmid N\ell(t)$, and therefore $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is not an integer. If $(t+3) \nmid N$, then $(t+3) \mid (t+2+N\cdot\ell(t))(t+1+N\cdot\ell(t))(t+N\cdot\ell(t))\cdots(1+N\cdot\ell(t))$; since $(t+2)(t+1) = a\cdot b\cdot(t+1)$, and $a\cdot b\cdot(t+1) \mid \lambda^*(t-1)\lambda^*(t)$, it follows that $\binom{t+2+N\cdot\ell(t)}{t+3}/N$ is an integer.

The result is stated in the following theorem.

Theorem 4.5 *For every natural number t with $t \geq 2$ let $\lambda(t) = \operatorname{lcm}(\binom{t}{m} \mid m = 1, 2, \ldots, t)$, $\lambda^*(t) = \operatorname{lcm}(1, 2, \ldots, t+1)$ and $\ell(t) = \prod_{i=1}^{t}\lambda(i)\cdot\lambda^*(i)$. Then*

(i) if $(t+3)$ and $(t+2)$ are composite, there exists an $LS[N](t, t+3, t+2+N\cdot\ell(t))$ for every $N \geq 2$,

(ii) *if* $(t+2)$ *is prime, there exists an* $LS[N](t, t+3, t+2+N\cdot\ell(t))$ *for any* $N \geq 2$ *with* $(t+2) \nmid N$,

(iii) *if* $(t+3)$ *is prime, there exists an* $LS[N](t, t+3, t+2+N\cdot\ell(t))$ *for any* $N \geq 2$ *with* $(t+3) \nmid N$.

Remark 4.1 The most celebrated theorem of Teirlinck is given in [31] stating that for given natural numbers t and v with $v \equiv t(\bmod\ (t+1)!^{(2t+1)})$, $v \geq t+1$, there is a large set of t-$(v, t+1, (t+1)!^{(2t+1)})$ designs. This is the first theorem proving the existence of non-trivial simple t-designs for all t. However, the values for v and λ are extremely large even for relatively small values of t. In [32], Teirlinck proves a much better result with drastically reduced values for λ, namely, there exists a large set of t-$(v, t+1, \ell(t))$ designs with $v \equiv t(\bmod\ \ell(t))$, as given above. To illustrate it, take for example $t = 5$. Then $(t+1)!^{(2t+1)} = (5+1)!^{11} = 26,956,124,946,896,309,452,800,000,000,000$, whereas $\ell(5) = 373,248,000$.

We consider another infinite series of $LS[N](4, 5, 4+20N)$ such that $\gcd(N, 30) = 1$ [32, 12]. Observe that the parameters of the 4-designs in these large sets are 4-$(4+20N, 5, 20)$, where $20 = \lambda_{\min}(4, 5, 4+20N)$. So, $LS[N](4, 5, 4+20N) = LS_{20}(4, 5, 4+20N)$. Now given an $LS[N](4, 5, 4+20N)$ it is straightforward to check that $N \mid \binom{5+20N}{6}$. Hence, by Corollary 3.2 there is an $LS[N](4, 6, 5+20N)$. Furthermore, when applying Corollary 3.2 to an $LS[N](4, 6, 5+20N)$ again, we can show that there is an $LS[N](4, 7, 6+20N)$, if in addition $7 \nmid N$. Its proof is straightforward and will be omitted.

Similarly, consider the large sets $LS_{60}(4, 5, 4+60N) = LS[N](4, 5, 4+60N)$ for all positive integers N with $\gcd(N, 60) = 1$ or 2, which are given in [32]. By applying Corollary 3.2 to these large sets, it is straightforward to verify that $N | \binom{5+60N}{6}$. Therefore it gives $LS[N](4, 6, 5+60N)$. When applying Corollary 3.2 to an $LS[N](4, 6, 5+60N)$ anew, we find that there is an $LS[N](4, 7, 6+60N)$, if in addition $7 \nmid N$.

We obtain the following result.

Theorem 4.6 *The following holds*

(i) *there is an* $LS[N](4, 6, 5+20N)$ *for any* N *with* $\gcd(N, 30) = 1$,

(ii) *there is an* $LS[N](4, 7, 6+20N)$ *for any* N *with* $\gcd(N, 210) = 1$,

(iii) *there is an* $LS[N](4, 6, 5+60N)$ *for any* N *with* $\gcd(N, 60) = 1$ *or* 2,

(iv) *there is an* $LS[N](4, 7, 6+60N)$ *for any* N *with* $\gcd(N, 60) = 1$ *or* 2 *and* $7 \nmid N$.

5 Conclusion

The paper introduces new recursive constructions for t-designs, s-resolvable t-designs including large sets of t-designs. The conditions required for the constructions are simply the necessary divisibility conditions for the considered designs, and the results turn out to be very effective. In particular, they reveal a remarkable link between the 'starting' and 'resulting' designs and appear to be significant to t-design theory. A full application of the results would certainly improve the number of known t-designs, s-resolvable t-designs and large sets of t-designs considerably.

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