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Minimal Cones and a Problem of Euler

Ulrich Dierkes (*)

Abstract – The minimal cones $C = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m; |x| \leq |y|\}$ are shown to minimize the weighted perimeter $P_\alpha(E) = \int |x|^\alpha |D\varphi_E|, \alpha \in \mathbb{R}, E \subset \mathbb{R}^m \times \mathbb{R}^m$, whenever $m + \alpha \geq 4$. This completes and improves recent results of Dierkes and Huisken [4].

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Introduction

Here we consider the n-dimensional analogue of a problem already investigated by Leonhard Euler [5], namely the variational integral

$$\mathcal{E}_\alpha(M) := \int_M |x|^\alpha d\mathcal{H}_{n-1} \quad , \alpha \in \mathbb{R},$$

where $M \subset \mathbb{R}^n$ denotes some smooth hypersurface and $\mathcal{H}_k$ stands for the k-dimensional Hausdorff measure. M is called *"stationary with respect to $\mathcal{E}_\alpha$"* or simply *"α-stationary"*, if the first variation $\delta\mathcal{E}_\alpha(M, X)$ vanishes, and a stationary surface M is called *"α-stable"* if the second variation $\delta^2\mathcal{E}_\alpha(M, X)$ is nonnegative for suitable variations X of M. Standard computations show that M is α-stationary, iff the mean curvature $H(x)$ and the unit normal $\nu = \nu(x)$ of M at $x \in M$ respectively satisfy

$$H(x) = \alpha|x|^{-2}\langle x, \nu\rangle, \; x \neq 0,$$

(*) *Indirizzo dell'A.*: Faculty of Mathematics, University of Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany; ulrich.dierkes@uni-due.de

while in addition M is also α-stable, if for all $\xi \in C_c^1(M, \mathbb{R})$ we have

$$\int_M |x|^\alpha \left\{ \frac{2}{\alpha} H^2 + |A|^2 \right\} \xi^2 d\mathcal{H}_{n-1} \le \int_M |x|^\alpha \left\{ \alpha |x|^{-2} \xi^2 + |\nabla \xi|^2 \right\} d\mathcal{H}_{n-1} \tag{0.1}$$

where $|A|$ denotes the length of the second fundamental form A of M, cp. Proposition 1.1 and 1.4 in [4].
In particular, if $C =$ closure M is a cone in $\mathbb{R}^n$ with only singularity at zero and if $M := C - \{0\}$ is α-stationary, then C is called α-stable, if (0.1) holds for all $\xi \in C_c^1(M, \mathbb{R})$. Obviously, every area-minimal cone C in $\mathbb{R}^n$ (i.e. $H = 0$ on $C - \{0\}$) such that $0 \in C$ and $C - \{0\}$ is a regular hypersurface is α-stationary and we have the following stability result

Theorem. *([4]) Let $\alpha > 3 - n$ and suppose $C \subset \mathbb{R}^n$ is an α-stationary cone with vertex at the origin and such that $(n - 3 + \alpha)^2 \ge 4|x|^2|A|^2 - 4\alpha$. Then C is also α-stable.*

(Note that the dimension of the cone C here is $(n - 1)$ rather than n in the paper [4].)
In particular the cone over the Clifford torus $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ is stable for all $\alpha \ge 1$. (Here $n = 4$ and $|A|^2 = 2|x|^{-2}$). Furthermore the 7-dimensional cones over products of spheres $S^1 \times S^5$, $S^2 \times S^4$ and $S^3 \times S^3$ are all α-stable, whenever $\alpha \ge \sqrt{48} - 7$, cp. the list in Corollary 2.2 of [4].
Moreover it could be shown in Theorem 3.1 of [4] that all cones

$$C_m = \{(x_1, \ldots, x_{2m}) \in \mathbb{R}^m \times \mathbb{R}^m; x_1^2 + \ldots + x_m^2 \le x_{m+1}^2 + \ldots + x_{2m}^2\}$$

minimize the integral $\mathcal{E}_\alpha$ in a suitable sense, if $1 \le \alpha \le 2(m - 1)$. So in particular the cone over the Clifford torus $S^1 \times S^1$ minimizes $\mathcal{E}_\alpha$, if $1 \le \alpha \le 2$.
The proof of these results uses a Weierstraß-Schwarz foliation type of argument, as it was introduced in the celebrated paper by Bombieri, De Giorgi and Giusti [1]. In fact, under the prescribed conditions on α and m the cones can be embedded in a *"field"* or *"calibration"* consisting of α-stationary surfaces. On the other hand, it is known since long, that a much easier device, known as *"sub-calibration"* is applicable in the case of classical area-minimal cones and we refer to the papers of Lawson [6], Miranda [8], Massari-Miranda [7], Morgan [9], Davini [2] and in particular to De Philippis and Paolini [3] for more pertinent information.
We show here, that many of the α-stable cones can be "sub-calibrated" with respect to the functional $\mathcal{E}_\alpha$ and are hence also minimizers for $\mathcal{E}_\alpha$ in a very general sense. The simplified proof which will be presented here, follows the approach by De Philippis and Paolini [3] and hence we may omit some of the details, referring to their paper [3].

1. α-Subminimal Sets and Minimal Cones

Let $E \subset \mathbb{R}^n$ be a measurable and $\Omega \subset \mathbb{R}^n$ be an open set. We define the *"α-perimeter"* of E in Ω as

$$P_\alpha(E,\Omega) \coloneqq \sup\left\{ \int_E \operatorname{div}\{|x|^\alpha g\}dx; \quad g \in C_c^1(\Omega,\mathbb{R}^n) \text{ with } |g(x)| \le 1\right\}.$$

Note that $P_\alpha(E,\Omega)$ is well defined (and possibly infinite) for all $\alpha \in \mathbb{R}$ and measurable sets E. However we have

Proposition 1. *Let $\alpha + n > 1$ and suppose ∂E is of class C^2. Then $P_\alpha(E,\Omega) = \int_{\partial E\cap\Omega} |x|^\alpha d\mathcal{H}_{n-1} < \infty$, for every bounded open set $\Omega \subset \mathbb{R}^n$.*

Remark. Clearly, if $0 \notin \partial E$ and Ω is bounded, then $\int_{\partial E\cap\Omega} |x|^\alpha d\mathcal{H}_{n-1}$ is always finite, independet of the value of $\alpha \in \mathbb{R}$.

Proof. Let $g \in C_c^1(\Omega,\mathbb{R}^n)$, $|g(x)| \le 1$ be arbitrary, then

$$\operatorname{div}\big(|x|^\alpha g\big) = \alpha|x|^{\alpha-2}(x\cdot g) + |x|^\alpha \operatorname{div} g$$

which is a function of Lebesgue-class $L_1(\Omega)$, if $\alpha + n > 1$. Denoting by φ_E the characteristic function of E, we obtain for arbitrary $\varepsilon > 0$ and ball $B_\varepsilon = B_\varepsilon(0) \subset \mathbb{R}^n$ with center zero,

$$\begin{aligned}
&\int_E \operatorname{div}\big(|x|^\alpha g\big)dx = \int_\Omega \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx = \\
&\int_{\Omega - B_\varepsilon} \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx + \int_{B_\varepsilon} \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx = \\
&\int_{\partial E - B_\varepsilon} |x|^\alpha (g\cdot\nu)d\mathcal{H}_{n-1} + \int_{\partial B_\varepsilon \cap E} |x|^\alpha (g\cdot\nu_\varepsilon)d\mathcal{H}_{n-1} + \int_{B_\varepsilon} \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx,
\end{aligned}$$

where ν and ν_ε stand for the exterior - or interior unit normals of ∂E and ∂B_ε respectively. By assumption $\alpha + n > 1$, thus the last two integrals tend to zero, as $\varepsilon \searrow 0$, whence

$$P_\alpha(E,\Omega) \le \int_{\partial E} |x|^\alpha d\mathcal{H}_{n-1}.$$

On the other hand, since by assumption ν is of class C^1 on the boundary ∂E, we may extend $\nu = \nu(x)$ to some function $N \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ such that $|N(x)| \le 1$ for all $x \in \mathbb{R}^n$. Take some function $\eta \in C_c^1(\Omega,\mathbb{R})$ with $|\eta(x)| \le 1$ for all $x \in \Omega$ and put

$g := \eta N \in C_c^1(\Omega, \mathbb{R}^n)$, then we have for every $\varepsilon > 0$ the relation

$$\int_E \operatorname{div}\big(|x|^\alpha g\,\big)dx = \int_\Omega \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx =$$
$$\int_{\Omega - B_\varepsilon} \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx + \int_{B_\varepsilon} \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx =$$
$$\int_{\partial E - B_\varepsilon} |x|^\alpha \eta d\mathcal{H}_{n-1} + \int_{\partial B_\varepsilon \cap E} |x|^\alpha \eta (N \cdot \nu_\varepsilon) d\mathcal{H}_{n-1} + \int_{B_\varepsilon} \varphi_E \operatorname{div}\big(|x|^\alpha g\big)dx.$$

By virtue of $\alpha + n > 1$ and upon letting $\varepsilon \searrow 0$ we find

$$\int_E \operatorname{div}\big(|x|^\alpha g\big)dx = \int_{\partial E} |x|^\alpha \eta d\mathcal{H}_{n-1},$$

whence also

$$P_\alpha(E, \Omega) \geq \int_{\partial E} |x|^\alpha d\mathcal{H}_{n-1}.$$

To see the finiteness of both integrals we assume w.l.o.g. that - locally near zero - ∂E is described by some C^2-function $x_n = \psi(x_1, \dots, x_{n-1})$ and that $\alpha < 0$. Then we have

$$\int_{\partial E \cap B_R(0)} |x|^\alpha d\mathcal{H}_{n-1} \leq \int_{B_R(0)} \big(x_1^2 + \dots + x_{n-1}^2\big)^{\alpha/2} \sqrt{1 + |D\psi|^2} dx_1 \dots dx_{n-1} \leq$$
$$\leq \text{const} \int_0^R r^{\alpha+n-2} dr < \infty, \text{ since } \alpha + n > 1.$$

■

Definition 1. *A measurable set $E \subset \mathbb{R}^n$ is a "local minimum for $P_\alpha(\,\cdot\,)$" in $\Omega \subset \mathbb{R}^n$, or simply "$\alpha$-minimal in Ω", if for all open, boundet sets $A \subset \Omega$*

$$P_\alpha(E, A) \leq P_\alpha(F, A)$$

for every measurable set F, such that the symmetric difference $E \Delta F \subset\subset A$ (i.e. the closure $\overline{E \Delta F} \subset A$).
E is called "α-subminimal in Ω", if for all bounded open $A \subset \Omega$ we have

$$P_\alpha(E, A) \leq P_\alpha(F, A)$$

for every measurable set $F \subset E$, such that $E \setminus F \subset\subset A$.

Theorem 1. *The cones $C_m = \big\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m; |x| \leq |y|\big\}$ are α-minimal in $\mathbb{R}^n$, $n = 2m$, provided $m + \alpha \geq 4$.*

REMARK. Combining Theorem 3.1 of [4] and Theorem 1 we conclude that the cone

$$C_2 = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2; |x| \le |y|\}$$

has least α-perimeter P_α in $\mathbb{R}^4$ for all $\alpha \ge 1$, in other words, the cone over the Clifford torus $S^1 \times S^1$ in $\mathbb{R}^4$ minimizes $\mathcal{E}_\alpha$ for any $\alpha \ge 1$. Conversely, there are no non-trivial stable cones in $\mathbb{R}^4$ with vertex at zero for every $\alpha < 1$, according to Theorem 2.3 in [4].

For the proof of the Theorem we present three Propositions, which follow immediately from [3].

PROPOSITION 2. *Let $\alpha + n > 1$ and suppose $E \subset \mathbb{R}^n$ and its complement $E^c := \Omega \setminus E$ are both α-subminimal in Ω, then E is α-minimal in Ω.*

PROOF. We first claim that, since $\alpha + n > 1$, it follows that $P_\alpha(E,\Omega) = P_\alpha(E^c,\Omega)$. Indeed, arguing as in Proposition 1 and because of $\operatorname{div}(|x|^\alpha g) \in L_1(\Omega)$ for any $g \in C_c^1(\Omega, \mathbb{R}^n)$, $|g(x)| \le 1$, $\alpha + n > 1$, we obtain that

$$\int_\Omega \operatorname{div}(|x|^\alpha g)\,dx = 0.$$

Therefore

$$\int_\Omega \varphi_E \operatorname{div}(|x|^\alpha g)\,dx = \int_\Omega (\varphi_E - 1)\operatorname{div}(|x|^\alpha g)\,dx = -\int_\Omega \varphi_{E^c} \operatorname{div}(|x|^\alpha g)\,dx,$$

where φ_{E^c} stands for the characteristic function of the complement $E^c = \Omega \setminus E$. Hence we have equality $P_\alpha(E,\Omega) = P_\alpha(E^c,\Omega)$ and the proof can be completed as in Proposition 1.2 of the paper [3]. ∎

PROPOSITION 3. *Let $\alpha + n > 1$ and suppose $E_k \subset E$, $k \in \mathbb{N}$, are measurable, α-subminimal sets in $\Omega \subset \mathbb{R}^n$ for all $k \in \mathbb{N}$. In addition assume that $E_k \to E$ in $L_{1,\mathrm{loc}}(\Omega)$ as $k \to \infty$, then also E is α-subminimal in Ω.*

PROOF. We only show the lower-semicontinuity of the α-perimeter, since the rest of the proof follows as in Proposition 1.3 of [3] To this end suppose $\varphi_{E_k} \to \varphi_E$ in $L_{1,\mathrm{loc}}(\Omega)$.
For $g \in C_c^1(\Omega, \mathbb{R}^n)$, $|g(x)| \le 1$ and $\alpha + n > 1$ we first have $\operatorname{div}(|x|^\alpha g) \in L_1(\Omega)$ and

$$\int_\Omega (\varphi_{E_k} - \varphi_E)\operatorname{div}(|x|^\alpha g)\,dx = \int_{E_k - E} \operatorname{div}(|x|^\alpha g)\,dx - \int_{E - E_k} \operatorname{div}(|x|^\alpha g)\,dx \to 0,$$

as $k \to \infty$. Whence

$$\int_\Omega \varphi_E \operatorname{div}\left(|x|^\alpha g\right) dx = \lim_{k\to\infty} \int_\Omega \varphi_{E_k} \operatorname{div}\left(|x|^\alpha g\right) dx \leq$$
$$\leq \liminf_{k\to\infty} P_\alpha\left(E_k, \Omega\right),$$

and the semicontinuity of P_α follows. Now we can argue as in Proposition 1.3 of [3] We skip over the details. ∎

Definition 2. *Let $\Omega \subset \mathbb{R}^n - \{0\}$ be open and $E \subset \Omega$ be measurable with boundary ∂E of class C^2. A vectorfield $\xi \in C^1\left(\Omega, \mathbb{R}^n\right)$ is an "α-subcalibration" of E (or ∂E) in Ω, if we have*

i) *For all $x \in \partial E, \xi(x) = \nu(x)$= exterior unit normal of ∂E at x.*
ii) $\operatorname{div}\left(|x|^\alpha \xi(x)\right) \leq 0$ *for all $x \in E \cap \Omega$.*
iii) $|\xi(x)| \leq 1$ *for all $x \in \Omega$.*

Proposition 4. *Suppose $E \subset \Omega$ has boundary of class C^2 and admits an α-subcalibration in $\Omega \subset R^n - \{0\}$. Then E is α-subminimal in Ω, i.e. $P_\alpha(E, A) \leq P_\alpha(F, A)$ for every bounded, open set $A \subset \Omega$ and all measurable $F \subset E$ with $E \setminus F \subset\subset A$.*

Proof. Analogous to Theorem 1.5 in [3] with perimeter replaced by α-perimeter. ∎

We now turn to the

Proof of Theorem 1. The idea is to approximate the cone C_m with a sequence of smooth sets E_k which admit α-subcalibrations. By Propositions 4 and 3 it follows that E itself is α-subminimal. Again the same reasoning applies to the complement $C_m^c = \mathbb{R}^n - C_m$ and hence the cone C_m is α-minimal by Proposition 2.
To this end consider the function $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}, (x, y) \mapsto f(x, y) := \frac{|x|^4 - |y|^4}{4}$, $x, y \in \mathbb{R}^m$. Obviously

$$C_m = \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m; \quad f(x, y) \leq 0\right\}$$

and for every $k \in \mathbb{N}$ we put

$$E_k := \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m; f(x, y) \leq -1/k\right\}$$

and

$$F_k := \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m; f(x, y) \leq 1/k\right\}.$$

Clearly, we have for all $k \in \mathbb{N}$

$$E_k \subset C_m \subset F_k \qquad \text{and}$$

$E_k \longrightarrow C_m, F_k \longrightarrow C_m$ both in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ as $k \longrightarrow \infty$ and the complement $F_k^c = \mathbb{R}^n \setminus F_k \subset C_m^c$ and $F_k^c \longrightarrow \mathbb{R}^n - C_m = C_m^c$ in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ as $k \longrightarrow \infty$.
Additionally, all boundaries in $\partial E_k, \partial F_k^c$ are smooth hypersurfaces in $\mathbb{R}^n$ and are all α-subminimal in $\mathbb{R}^n$. Indeed, we claim that the vectorfield

$$\xi(x,y) := \pm \frac{Df(x,y)}{|Df(x,y)|}$$

defines an α-subcalibration for both E_k and F_k^c in $\mathbb{R}^n - \{0\}$ respectively, i.e. we have i), ii) and iii) of Definition 2 for all E_k and F_k^c, $k \in \mathbb{N}$.
While i) and iii) are obviously fulfilled in both cases, ii) needs a simple calculation: If $f(x,y) = \frac{|x|^4 - |y|^4}{4}$ we find successively

$$\begin{aligned} f_x &= |x|^2 x, f_y = -|y|^2 y, \quad |Df|^2 = |x|^6 + |y|^6 \\ f_{x_i x_j} &= 2x_i x_j + \delta_{ij}|x|^2, f_{y_i y_j} = -2y_i y_j - \delta_{ij}|y|^2, \\ f_{x_i y_j} &= 0 \end{aligned}$$

for all indices $i, j = 1, \ldots, m$. Also, for $(x,y) \neq (0,0)$ we have $|(x,y)|^\alpha = \left(|x|^2 + |y|^2\right)^{\alpha/2}, D|(x,y)|^\alpha = \alpha|(x,y)|^{\alpha-2}(x,y)$, $\frac{Df}{|Df|} = \frac{\left(|x|^2 x, -|y|^2 y\right)}{\left(|x|^6 + |y|^6\right)^{1/2}}$ and

$$|Df|^3 \mathrm{div}\Big(\frac{Df}{|Df|}\Big) = \left(|x|^4 - |y|^4\right)\left\{(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4\right\},$$

whence

$$\begin{aligned} &|Df|^3 \mathrm{div}\Big[|(x,y)|^\alpha \frac{Df}{|Df|}\Big] = \\ &\quad \alpha|Df|^2|(x,y)|^{\alpha-2}\left[|x|^4 - |y|^4\right] + \left(|x|^2 + |y|^2\right)^{\alpha/2} \cdot \\ &\qquad \left\{\left(|x|^4 - |y|^4\right)\left[(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4\right]\right\} = \\ &\quad \alpha|(x,y)|^{\alpha-2}\left[|x|^4 - |y|^4\right]\left(|x|^6 + |y|^6\right) \\ &\qquad + |(x,y)|^\alpha\left(|x|^4 - |y|^4\right)\left[(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4\right] = \\ &\quad |(x,y)|^{\alpha-2}\left(|x|^4 - |y|^4\right)\left\{(m-1+\alpha)|x|^6 - 3|x|^4|y|^2 - 3|x|^2|y|^4 + (m-1+\alpha)|y|^6\right\}. \end{aligned}$$

Upon putting $t := \frac{|x|^2}{|y|^2}$ we see that the sign of $\mathrm{div}\left[|(x,y)|^\alpha \frac{Df}{|Df|}\right]$ is the same as of $f(x,y) = \frac{|x|^4 - |y|^4}{4}$ provided the polynomial

$$p_{m,\alpha}(t) := (m-1+\alpha)t^3 - 3t^2 - 3t + (m-1+\alpha)$$

is nonnegative for all $t \geq 0$. Since for every $t \geq 0$ we have $3t^3 - 3t^2 - 3t + 3 \geq 0$ (with zero minimum for $t \geq 0$ at $t = 1$) it also follows that $p_{m,\alpha}(t) \geq 0$ for all $t \geq 0$, if we assume that $(m - 1 + \alpha) \geq 3$.
Concluding we have shown that the vectorfield $\xi(x, y) = \frac{Df(x,y)}{|Df(x,y)|}$ defines an α-subcalibration for the sets E_k in $\mathbb{R}^n - \{0\}$, $k \in \mathbb{N}$, while $-\xi(x, y)$ furnishes an α-subcalibration for the sets $F_k^c = \mathbb{R}^n - F_k$ in $\mathbb{R}^n - \{0\}$. Proposition 4 yields the α-subminimality of the sets E_k and F_k^c in $\mathbb{R}^n - \{0\}$ respectively. However, $0 \notin E_k$ or F_k^c and we have $P_\alpha(E_k, A) = P_\alpha(E_k, A - \{0\}) \leq P_\alpha(F, A - \{0\}) \leq P_\alpha(F, A)$ for every bounded open set $A \subset \mathbb{R}^n$ and arbitrary $F \subset E_k$ with $E_k \setminus F \subset\subset A$, which shows that all sets E_k are also α-subminimal in all of $\mathbb{R}^n$ (rather that just in $\mathbb{R}^n - \{0\}$). A similar argument implies the α-subminimality of F_k^c in $\mathbb{R}^n$ and since $E_k \to C_m$ or $F_k^c \to \mathbb{R}^n - C_m$ both in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ as $k \to \infty$, we infer from Proposition 3 that C_m as well as its complement $C_m^c = \mathbb{R}^n - C_m$ are both α-subminimal in $\mathbb{R}^n$. An application of Proposition 2 concludes the proof of the Theorem 1. ∎

Remark. Using the same type of argument as in the proof of Theorem 3.1 one can also deal with the minimal cones

$$C_{m,k} = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^k; \quad (k - 1)|x|^2 \leq (m - 1)|y|^2\}.$$

A subcalibration might then be determined by the normalized gradient of the function

$$f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}, f(x, y) := \frac{1}{4}\left((k - 1)^2|x|^4 - (m - 1)^2|y|^4\right)$$

under suitable conditions on m and k, however we shall not dwell on this.

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